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Analysis on the Hesitation and its Application to Decision Making

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ABSTRACT

A novel score function based on the Poincaré metric is proposed and applied to a decision-making problem. Decision-making on Fuzzy Sets (FSs) has been considered due to the flexibility of the data, and it is applied to the decision-making. However, decisions with FSs are sometimes nondecisive even for different membership degrees. Hence, Intuitionistic Fuzzy Sets (IFSs) data is applied to design a score function for the decision-making with the Poincaré metric. This function is supported by the profound information of IFSs; IFSs include hesitation degree together with membership and non-membership degree. Hence, IFS membership and non-membership degree are expressed as two-dimensional vectors satisfying the Poincaré metric for simplification. At the same time, the proposed approach addresses the hesitation information in the IFS data. Next, a score function is proposed, constructed and provided. The proposed score function has a strict monotonic property and addresses the preference without resorting to the accuracy function. The strict monotonic property guarantees the preference of all attributes. Additionally, the existing problem of score function design in IFSs is addressed: they return zero scores even with different meanings for the same membership and non-membership degree. The advantages of the proposed score function over existing ones are demonstrated through illustrative examples. From the calculation results, the proposed decision score function discriminates between all candidates. Hence, the proposed research provides a solid foundation for the hesitation analysis on the decision-making problem.

1. Introduction

Decision-making has been performed by fundamentally considering the data as fuzzy sets in many existing problems [1-6]. Specifically, problems in the decision-making of multiple criteria have been resolved using fuzzy sets (FSs) and intuitionistic fuzzy sets (IFSs) [7, 8]. Xuezhen et al.

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performed decision-making with the help of the Choquet integral [9]. However, fuzzy data still seem challenging if they are considered with IFS structures. Even if it is challenging, IFSs effectively describe the decision circumstance through three degrees, support, descent, absence as the membership, non-membership, and hesitation degree, respectively; $\pi(x)$, $\mu(x)$ and $\nu(x)$ for x over the universe of discourse.

The score function is considered to decide the preference for specific attributes according to whether it is a single or multicriteria problem [1,7,8,10]. However, a lack of hesitation makes the decision problem rather strict due to dividing $\mu(x)$ and $\nu(x)$, even when the decision measure was proposed [1,7]. As a more general type of FS, Liu et al. and Wang et al. treated the data as IFSs [8, 10], but there was not much research illustrating the decision-making problem with IFS data because of unclear because of unsure definitions of $\pi(x)$ and atypical $\mu(x)$ and $\nu(x)$ degrees. The hesitation degree is derived from the combination of $\mu(x)$ and $\nu(x)$ as $\pi(x)$ in IFSs by Atanassov [11], and its distance measure was also proposed by Mahanta and Panda [12]. Previous research has summarized the distance and similarity measures of IFSs [13, 14]. These distance measurements mostly consider IFSs in one-dimensional space. It is believed that when the data are projected into a higher dimension, the differences between data with slight changes will be more discriminative [15]. Therefore, the scores obtained by the score function between $\mu(x)$ and $\nu(x)$ will show more differences when projecting IFSs into higher dimensions [3].

In the existing research, the absence of the $\nu(x)$ or $\pi(x)$ in the score function sometimes showed no effect on decision results in [1, 8]. In this regard, Ye insisted that the hesitation degree can positively or negatively affect the score function in research [16]. Wang et al. empirically proposed positive and negative influences on the score function with hesitation [10], which depended on the value of $\mu(x) - \nu(x)$. Recently, Gao et al. proposed the score function to project the $\pi(x)$, $\mu(x)$ and $\nu(x)$ degrees into a higher dimension similar to the form of the Gaussian kernel function [3]. Developing score functions for decision-making problems using IFSs shows the advantages of projecting the elements characterizing IFSs into higher dimensional space. However, a narrow range of scores may not be able to handle the slight changes in the membership and non-membership degrees well. In addition, decision-making strategies are applied to uncertain data in fuzzy environments, such as TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) and TODIM (an acronym in Portuguese for Interactive and Multicriteria Decision-Making) [17-20].

By summarizing the existing research results, the following problems are addressed in this study:

i. Existing studies have preference functions with only two $\mu(x)$, and $\nu(x)$ over the one-dimensional space [1,8,10,16]
ii. Existing studies provide a very narrow score range that is not easily differentiated, e.g., $[-1,1]$ or $[e^{-1}, e]$ [3]

Satisfactory decision results could be achieved if only small data are used, but this does not meet practical requirements. In the research, the preference result was not decisive, so an additional judgement function was needed: the accuracy function [7]. In this regard, we proposed a novel preference function based on the Poincaré metric. The Poincaré metric has shown advantages on uncertain data [20]. Previous score function designs for IFSs have considered the difference between two comparable $\mu(x)$ and $\nu(x)$ over one-dimensional space, and the decision results sometimes showed unclear results; the same preference value was even different $\mu(x)$ and $\nu(x)$ because of the preference measure structure. The Poincaré metric is the proper tool to address IFSs, which are expressed with two independent vectors: $\mu(x)$ and $\nu(x)$.

With respect to the preference, explicit value on ordering can be accomplished through a score function. For a clearer decision, a strict increase/decrease score function is necessary. The following procedure is the design of the score function. First, we investigate the design of the score function, which is designed to satisfy strict monotonic properties from the preference function. It includes the relation between the score function of IFSs and its strict monotonicity with respect to their $\mu(x)$ and $\nu(x)$. Next, it is proved by theorem; that is, the score function shows an exponentially increasing rate according to the characteristic of the Poincaré metric [21]. The obtained result is applied to the multicriteria decision-making problem, and $\mu(x)$ and $\nu(x)$ are considered independent variables. To obtain significant results from the proposed score function, Spearman's rank correlation coefficient is applied to the scores when the ranking order is different from that in previous research [22]. Spearman's rank correlation also shows advantages of fuzzy theory [23].

The existing studies are compared with the proposed score function calculation, which are illustrated in examples. The obtained calculation results overcome the existing problem; when the score is not discriminative, an additional accuracy function is needed [7]. From the comparisons with the existing research, the proposed score function shows it approaches from ∞ to $-\infty$; rather narrow and small score in previous research [1,3,8,10]. The results provide a higher discrimination score when they are applied to comparable IFSs. Hesitations are illustrated via Figure 1, which is perturbed in $\mu(x)$ and $\nu(x)$; the score function with respect to $\mu(x)-\nu(x)$ and $\pi(x)$ is also illustrated in Figure 4. The proposed score function's strict monocity is proven and illustrated in Figures 3-4 as well.

This paper is organized as follows. In Section 2, the Poincaré metric and score function including the existing results are briefly introduced. In Section 3, considering $\mu(x)$ and $\nu(x)$ as the vector form, a score function for single and multicriteria structures is proposed, and its strictly monotonic property is provided. The score function is designed based on the Poincaré metric, as analysed in Section 2. The relation with $\pi(x)$ and the difference between $\mu(x)$ and $\nu(x)$ is also illustrated graphically. In Section 4, three illustrative examples are presented: multicriteria decision-making problems. Examples show that the proposed score function overcomes the no decision cases in previous research. Other examples illustrate multicriteria decision-making problems, which show more significant results. The results are discussed in Section 5. Finally, conclusions are included in Section 6.

2. Preliminary

In this section, the concept of the Poincaré metric is discussed to formulate measure realization. In addition, the properties of score functions and their relation to decision-making strategies are also illustrated with IFSs.

2.1 Poincaré Metric

The Poincaré metric and the distance in the Poincaré ball have been emphasized in machine learning [21, 24-27]. The Poincaré metric definition is provided in Definition 1.

Definition 1 [26]. The Poincaré metric of the distance for a point in $\{P=(x,y)|\ y>0\}$ in the hyperbolic plane is defined as:

$$ds^2=\frac{dx^2+dy^2}{y^2} \tag{1}$$

where dx and dy denote the difference in the distance in the x and y axes of the two-dimensional space, respectively.

By Definition 1, the distance between two arbitrary points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ over the hyperbolic plane can be defined as noted in [12]:

$$d(P_1, P_2) = \text{acosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2}\right) \quad (2)$$

where $acosh$ denotes the arc hyperbolic cosine function.

Definition 2 [21]. The Poincaré distance between two n-dimensional points $MP_1 = (x_1, x_2, \dots, x_n)$ and $MP_2 = (y_1, y_2, \dots, y_n)$, whose l_2 norm is less than 1, in a unit Poincaré ball can be derived as:

$$d_m(MP_1, MP_2) = arcosh(1 + \delta(MP_1, MP_2)) \quad (3)$$

where $\delta(MP_1, MP_2)$ is an isometric invariant and expressed as:

$$\delta(a, b) = \frac{\|MP_1 - MP_2\|^2}{2(1 - \|MP_1\|^2)(1 - \|MP_2\|^2)} \quad (4)$$

where $\|\cdot\|$ denotes the l_2 norm.

The Poincaré metric and distance are measures of points in a hyperbolic space, and ensure the value of the distance between two points is within the domain $[0, \infty)$ [21]. There is an exponentially increasing rate for the Poincaré distance, according to Eq. (1) and Eq. (3). This means that there is a larger difference when the variables have a slight change. The conventional score functions of IFSs can be considered a one-dimensional distance, and their increasing rate is primarily constant or linear.

2.2. Intuitionistic Fuzzy Sets

With the information of intuitionistic fuzzy sets (IFSs), an IFS A has the structure with

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \quad (5)$$

where X is the universe of the discourse, $\mu_A: X \to [0,1]$, $\nu_A: X \to [0,1]$, with the constraint $0 \le \mu_A(x) + \nu_A(x) \le 1$. $\mu_A(x)$ and $\nu_A(x)$ denote the membership and non-membership degrees of x to A, respectively. IFS research applications have been carried out in control, data analysis, and other human-centric problems [12, 28-32].

Then the hesitation degree $\pi_A(x)$ of x to A is defined as:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \quad (6)$$

where $0 \le \pi_A(x) \le 1, \forall x \in X$.

IFSs $\mu_A(x)$ and $\nu_A(x)$ are also characterized as follows.

Definition 3 [7]. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ be an IFS over X; then, there is always a framework of intervals for membership and non-membership degrees satisfying $[\mu_A(x), 1 - \nu_A(x)]$ and $[\nu_A(x), 1 - \mu_A(x)]$, respectively.

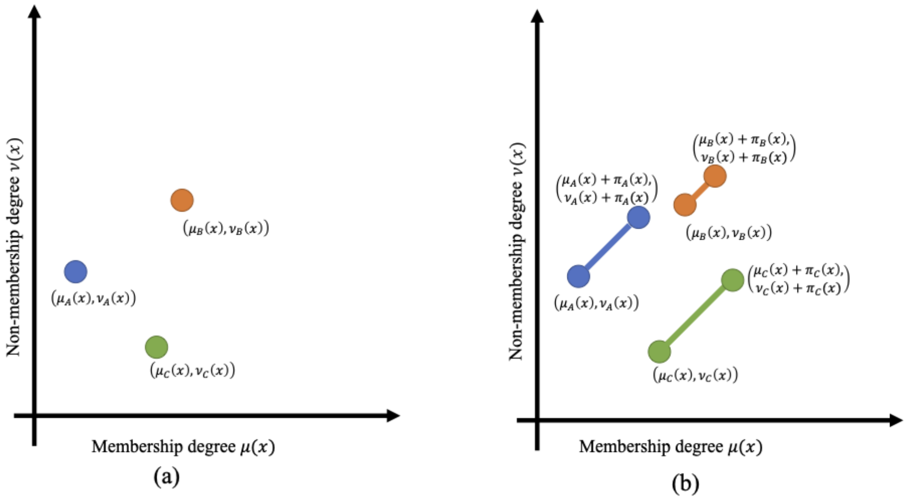

Fig. 1. (a) IFSs in the Cartesian Plane, (b) the intervals of IFSs in the Cartesian plane

According to the hesitation degree, $1-\nu_A(x) \geq \mu_A(x)$ and $1-\mu_A(x) \geq \nu_A(x)$ are always satisfied. Meanwhile, due to Eq. (6), $1-\nu_A(x)$ and $1-\mu_A(x)$ can be derived as $\mu_A(x)+\pi_A(x)$ and $\nu_A(x)+\pi_A(x)$, respectively. When describing the IFSs in a Cartesian plane, $\mu_A(x)$ and $\nu_A(x)$ are considered to describe the IFS, and the IFS can be illustrated as the points in Figure 1 (a). Considering Definition 3, Figure 1 (b) displays the intervals of different IFSs. However, the Cartesian plane shown in Figure 1 is the metric space for different IFSs. The score function is not the metric between different IFSs but the metric between $\mu_A(x)$ and $\nu_A(x)$ in the same IFS.

2.3. Score Functions

For A and B in IFSs, existing score functions and decision-making strategies are recalled, and the operator '$\prec$' indicates the preference of different IFSs. $A \prec B$ indicates that the score of A is less than that of B, which means we prefer B over A.

The score function $S_C(A)$ was proposed by Chen and Tan [1], and it was expressed as follows:

$$S_C(A) = \mu_A(x) - \nu_A(x) \tag{7}$$

where $S_C(A)$ exists over $[-1,1]$. When $S(A) < S(B)$ is satisfied, it is indicated by the preference $A \prec B$.

However, Eq. (7) faces difficulty in decision-making when $S(A) = S(B)$. To overcome this difficulty, an accuracy function $H(A)$ was proposed by [7]:

$$H(A) = \mu_A(x) + \nu_A(x) \tag{8}$$

where $H(A) \in [0,1]$.

A large $H(A)$ implies that more information is clarified, which can then be used to supplement decision-making. However, it is still insufficient to consider the hesitation property [3].

A score function $S_L(A)$ by Liu and Wang was also proposed by the consideration of hesitation [8]:

$$S_L(A) = \mu_A(x) + \mu_A(x)\pi_A(x). \tag{9}$$

The same relation with preference and score function is satisfied in Eq. (9). However, abstaining from a non-membership degree in Eq. (9) leads to conflict in zero membership degree.

Wang et al. categorized a score function $S_W(A)$ by considering the cross entropy of IFSs and intuitionistic fuzzy numbers (IFNs) [10]:

$$S_W(A) = \begin{cases} \mu_A(x) - \nu_a(x) + E(x)\pi_A(x), & \mu_A(x) > \nu_A(x) \\ \mu_A(x) - \nu_A(x) - E(x)\pi_A(x), & \mu_A(x) < \nu_A(x) \\ \quad 0, & \mu_A(x) = \nu \end{cases} \tag{10}$$

where $E(x)$ denotes the cross-entropy. However, it has the problem in dealing with IFSs in particular cases, which has been illustrated in the conclusions of [3].

Gao et al. proposed a score function with three degrees, $\mu_A(x)$, $\nu_A(x)$ and $\pi_A(x)$ [3]:

$$S_G(A) = \frac{e^{\mu_A(x) - \nu_A(x) + \pi_A(x)\left(\mu_A(x) - \nu_A(x)\right)^3}}{1 + \pi_A(x)} \tag{11}$$

However, it is not effective when $\mu_A(x)$ and $\nu_A(x)$ are similar: S_G sometimes requires a precision up to more than 10^{-6}.

3. Score Function with Poincaré Metric

Most of the existing score functions of IFSs consider the difference between $\mu(x)$ of comparable attributes and $\nu_A(x)$ over the one-dimensional space. Since the Poincaré metric is a metric for the elements in hyperbolic space, we introduce the framework of IFSs expressed in a vector form satisfying the Poincaré metric.

3.1 Vector Structure on IFSs

This subsection provides two ways to establish IFSs in a vector structure for multicriteria alternatives over the hyperbolic space. For all IFS, there always exist transformations: $\mu_A(x) \rightarrow [\mu_A(x), 1 - \nu_A(x)]$ and $\nu_A(x) \rightarrow [\nu_A(x), 1 - \mu_A(x)]$ according to Definition 3 [7]. By this consideration and the first expression of the hesitation degree $\pi_A(x)$ in Eq. (6), we can express IFS A with a two-dimensional vector framework.

Definition 4. Let $A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle | x_i \in X\}$ be IFS over the universe of the discourse X. Then A is expressed as a set of two-dimensional vectors:

$$A^*(x_i) = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle | x_i \in X\} \tag{12}$$

where $\mu_A(x_i)$ and $\nu_A(x_i)$ are denoted by $(\mu_A(x_i), \mu_A(x_i) + \pi_A(x_i))$ and $(\nu_A(x_i), \nu_A(x_i) + \pi_A(x_i))$, respectively, and x_i denotes the i^{th} elements in X.

In Eq. (12), A^* includes the component with two independent elements $\mu_A(x)$ and $\nu_A(x)$ for $x \in X$. Hence, the $\mu_A(x)$ and $\nu_A(x)$ bounds are denoted as $(\mu_A(x_i), \mu_A(x_i) + \pi_A(x_i))$ and $(\nu_A(x_i), \nu_A(x_i) + \pi_A(x_i))$. Figure 2 (a) shows the intervals of the IFSs, and (b) displays the projections of the IFSs from (a) to an orthogonal space. The points in the same colour are $\mu_A(x)$ and $\nu_A(x)$ in the same IFSs. Then, the Poincaré metric can be considered the metric for the score function.

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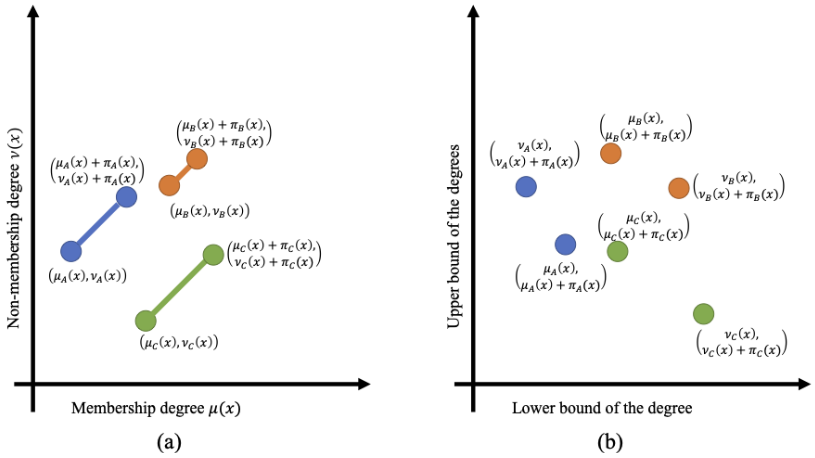

Fig. 2. (a) Intervals of the different IFSs, (b) projections of the IFSs in (a) to an orthogonal space

To resolve the multicriteria decision problem, each element in the universe of discourse is considered as each criterion. This means that Definition 4 separates the multicriteria IFS into different vector form IFSs depending on different criteria, and it makes it possible to map the values of $\mu_A(x)$ and $\nu_A(x)$ to the two-dimensional space. Based on the knowledge of hesitation, $\mu_A(x)+\pi_A(x)\geq 0$ and $\nu_A(x)+\pi_A(x)\geq 0$ is always satisfied. However, we should consider certain situations that satisfy $\mu_A(x)=1$ or $\nu_A(x)=1$. In these cases, the vector form IFSs are expressed as $\langle(1,0),(0,1)\rangle$ and $\langle(0,1),(1,0)\rangle$, which are not included in the Poincaré half plan. However, the score function based on the Poincaré distance can still satisfy these two situations, and further discussion will be elaborated after the definition of the score function.

By Eq. (12), IFSs can be applied to the Poincaré metric and illustrated in the hyperbolic space. The function of the novel score function on the IFS is defined based on Eq. (1). The summation of scores obtained by different single-criterion IFSs can be applied to multicriteria decision-making problems.

It should be noted that Definition 4 still considers the multicriteria problem with IFS in multidimensional vector form with IFSs, and the detailed vector form of IFS A can be expressed as:

$$A=\left\{\begin{matrix}\langle x_1,(\mu_A(x_1),\mu_A(x_1)+\pi_A(x_1)),(\nu_A(x_1),\nu_A(x_1)+\pi_A(x_1))\rangle,\\ \langle x_2,(\mu_A(x_2),\mu_A(x_2)+\pi_A(x_2)),(\nu_A(x_2),\nu_A(x_2)+\pi_A(x_2))\rangle,\\ \ldots,\langle x_n,(\mu_A(x_n),\mu_A(x_n)+\pi_A(x_n)),(\nu_A(x_n)+\pi_A(x_n))\rangle\end{matrix}\middle| x_i\in X\right\} \quad (13)$$

where n is the number of criteria and X is the offset of all criteria for consideration. The score is calculated independently for each element x_i in this vector form.

However, the multicriteria IFS can also be summarized as two vectors in a multidimensional space; $\mu_A(x)$ and $\nu_A(x)$ of different elements x_i are defined holistically in vectors, as in Definition 5.

Definition 5. Let $A=\{\langle x_i,\mu_A(x_i),\nu_A(x_i)\rangle | x_i\in X\}$ be IFSs over the universe of discourse X. Then A can be expressed as a set of multi-dimensional vectors:

$$A_m^*=\{\langle x_i,(\mu_A(x_1),\mu_A(x_2),\ldots,\mu_A(x_n)),(\nu_A(x_1),\nu_A(x_2),\ldots,\nu_A(x_n))\rangle | x_i\in X\} \quad (14)$$

where n is the number of criteria and $n \geq 2$. For simplicity, we can denote $\left(\mu_A(x_1), \mu_A(x_2), \dots, \mu_A(x_n)\right)$ and $\left(\nu_A(x_1), \nu_A(x_2), \dots, \nu_A(x_n)\right)$ as $M_A(x)$ and $N_A(x)$, respectively. Then, IFS A can be rewritten as $A_m^*(x) = \{\langle x, M_A(x), N_A(x)\rangle | x_i \in X\}$.

Definition 5 considers the multicriteria IFSs in two parts: $\mu_A(x)$ and $\nu_A(x)$. In Eq. (14), the membership degrees of different criteria are considered holistic independent variables. Similarly, non-membership degrees are considered another holistic independent variable including different criteria. We can calculate the difference between the membership and non-membership degrees of different criteria uniformly and holistically in the score function instead of calculating the difference for different criteria independently. Meanwhile, since we apply the Poincaré distance to the vector-form IFSs, it should avoid zero denominators in the score function. The IFSs in a multidimensional vector form make the constraint more flexible, and it only requires the l_2 norm of $M_A(x)$ and $N_A(x)$ to not be equal to one.

3.2. Preference Score Functions

Based on the different vector forms of the IFSs, two novel score functions are proposed. Their properties are also illustrated in this subsection.

3.2.1. Preference score function for single-criterion vector form and its property

The novel score function is defined in single-criterion vector form based on Definition 4 and Eq. (4).

Definition 6. For IFS A, the score function $S_P\left(A(x)\right)$ on the single-criterion decision problem is expressed as:

$$S_P = \begin{cases} acosh\left(1 + \phi\left(\mu_A(x), \nu_A(x)\right)\right), \text{for } \mu_A(x) \geq \nu_A(x) \\ acosh\left(1 - \phi\left(\mu_A(x), \nu_A(x)\right)\right), \text{for } \mu_A(x) < \nu_A(x) \end{cases} \tag{15}$$

where $acosh$ denotes the arc hyperbolic cosine function, and $\phi(\mu_A(x), \nu_A(x))$ denotes:

$$\phi\left(\mu_A(x), \nu_A(x)\right) = \frac{\|\mu_A(x) - \nu_A(x)\|}{2\left(\mu_A(x) + \pi_A(x)\right)\left(\nu_A(x) + \pi_A(x)\right)} \tag{16}$$

Here, we discuss the special conditions mentioned in Subsection 3.1. Under the condition $\mu_A(x) = 1$, we obtain the single-criterion vector from IFS $A_1 = \{\langle(1,0), (0,1)\rangle\}$. The score after the calculation is $S_P(A_1) \to \infty$. For another condition $\nu_A(x) = 1$, the single-criterion vector from IFS is $A_2 = \{\langle(0,1), (1,0)\rangle\}$, and the score $S_P(A_2) \to -\infty$. Under these two conditions, the score function approaches the two unique values ∞ and $-\infty$, and they are the global maximum and minimum values of the score function.

In the multicriteria problem, the score function is represented according to Eq. (15) as follows:

$$S_P\left(A(x)\right) = \sum_{i=1}^{n} \omega_i S_P\left(A(x_i)\right) \tag{17}$$

where n is the number of criteria, ω_i denotes the weight of the i^{th} alternative, $\omega_i \in [0,1]$, and $\sum_{i=1}^{n} \omega_i = 1$.

From Definition 6, we conclude the preference relation $A \prec B$ when $S_P(A) < S_P(B)$ is satisfied. Since the score function depends on the terms $\mu_A(x)$ and $\nu_A(x)$, the term $\phi\left(\mu_A(x), \nu_A(x)\right)$ inside the arc hyperbolic cosine function can be derived as:

$$\phi\big(\mu_A(x),\nu_A(x)\big)=\frac{\|\mu_A(x)-\nu_A(x)\|}{2\big(\mu_A(x)+\pi_A(x)\big)\big(\nu_A(x)+\pi_A(x)\big)}$$

$$=\frac{\sqrt[2]{\big(\mu_A(x)-\nu_A(x)\big)^2+\big(\nu_A(x)-\mu_A(x)\big)^2}}{2\big(\mu_A(x)+\pi_A(x)\big)\big(\nu_A(x)+\pi_A(x)\big)}$$

$$=\frac{|\mu_A(x)-\nu_A(x)|}{\sqrt{2}\big(\mu_A(x)+\pi_A(x)\big)\big(\nu_A(x)+\pi_A(x)\big)}$$

where $|\cdot|$ denotes the absolute value, since the arc hyperbolic cosine function is strictly monotonic in its domain, $[1,\infty)$, and the strict monotonicity of $S_P\big(A(x)\big)$ depends on the strict monotonicity of $\phi\big(\mu_A(x),\nu_A(x)\big)$. For simplicity, we denote this as $\phi\big(A(x)\big)$ instead of $\phi\big(\mu_A(x),\nu_A(x)\big)$.

Now, we show the relation between the score function and $\mu_A(x)$ and $\nu_A(x)$. The result is intuitive for decision-making. Theorem 1 shows the score function $S_P\big(A(x)\big)$ dependence on $\mu_A(x)$ and $\nu_A(x)$.

Theorem 1. For an IFS A, the proposed score function $S_P\big(A(x)\big)$ strictly increases with respect to the $\mu_A(x)$ and strictly decreases with respect to the $\nu_A(x)$.

Proof. Since $acosh(\cdot)$ strictly increases over $[1,\infty)$, so the strict monotonicity of $\phi(A)$ in $S_P\big(A(x)\big)$ need to be verified. The proof of Theorem 1 is derived directly for the two cases.

1. For $\mu_A(x)\geq\nu_A(x)$, the partial derivative of $\phi\big(A(x)\big)$ is obtained as:

$$\frac{\partial\phi\big(A(x)\big)}{\partial\mu_A(x)}=\frac{\big(\mu_A(x)+\pi_A(x)\big)\big(\nu_A(x)+\pi_A(x)+\mu_A(x)-\nu_A(x)\big)}{\sqrt{2}\Big(\big(\mu_A(x)+\pi_A(x)\big)\big(\nu_A(x)+\pi_A(x)\big)\Big)^2}=\frac{1}{\sqrt{2}\big(\nu_A(x)+\pi_A(x)\big)^2}>0$$

$$\frac{\partial\phi\big(A(x)\big)}{\partial\nu_A(x)}=\frac{\big(\nu_A(x)+\pi_A(x)\big)\big(\mu_A(x)-\nu_A(x)-1+\nu_A(x)\big)}{\sqrt{2}\Big(\big(\mu_A(x)+\pi_A(x)\big)\big(\nu_A(x)+\pi_A(x)\big)\Big)^2}=-\frac{1}{\sqrt{2}\big(\mu_A(x)+\pi_A(x)\big)^2}<0$$

The score function $S_P\big(A(x)\big)$ strictly increases and decreases with respect to $\mu_A(x)$ and $\nu_A(x)$ since $\mu_A(x)$ and $\nu_A(x)$ are in [0,1].

2. For $\mu_A(x)<\nu_A(x)$, the partial derivative of $\phi(A)$ is derived as follows:

$$\frac{\partial\phi\big(A(x)\big)}{\partial\mu_A(x)}=\frac{\big(\mu_A(x)+\pi_A(x)\big)\big(-1+\mu_A(x)+\nu_A(x)-\mu_A(x)\big)}{\sqrt{2}\Big(\big(\mu_A(x)+\pi_A(x)\big)\big(\nu_A(x)+\pi_A(x)\big)\Big)^2}=\frac{-1}{\sqrt{2}\big(\nu_A(x)+\pi_A(x)\big)^2}<0$$

$$\frac{\partial\phi\big(A(x)\big)}{\partial\nu_A(x)}=\frac{\big(\nu_A(x)+\pi_A(x)\big)\big(\mu_A(x)+\pi_A(x)+\nu_A(x)-\mu_A(x)\big)}{\sqrt{2}\Big(\big(\mu_A(x)+\pi_A(x)\big)\big(\nu_A(x)+\pi_A(x)\big)\Big)^2}=\frac{1}{\sqrt{2}\big(\mu_A(x)+\pi_A(x)\big)^2}>0$$

In this case, $S_P\big(A(x)\big)$ shows opposite relation with respect to $\mu_A(x)$ and $\nu_A(x)$. Hence, the Theorem 1 is proved.

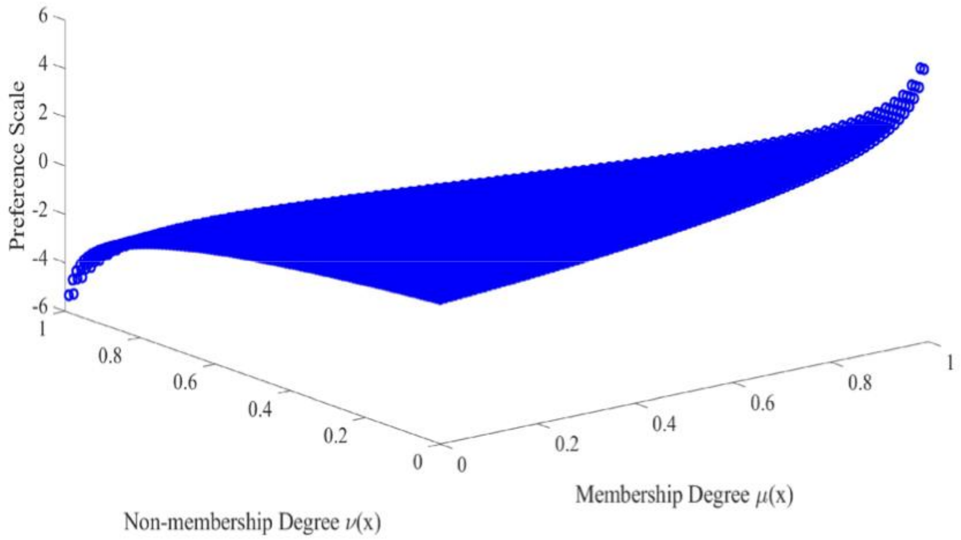

Fig.3. Preference Score with Membership Degree and Non-membership Degree

where $|\cdot|$ denotes the absolute value, since the arc hyperbolic cosine function is strictly monotonic in its domain, $[1,\infty)$, and the strict monotonicity of $S_P\big(A(x)\big)$ depends on the strict monotonicity of $\phi\big(\mu_A(x),\nu_A(x)\big)$. For simplicity, we denote this as $\phi\big(A(x)\big)$ instead of $\phi\big(\mu_A(x),\nu_A(x)\big)$.

Now, we show the relation between the score function and $\mu_A(x)$ and $\nu_A(x)$. The result is intuitive for decision-making. Theorem 1 shows the score function $S_P\big(A(x)\big)$ dependence on $\mu_A(x)$ and $\nu_A(x)$.

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Definition 7. For an IFS A, A is a positive alternative for $d_A(x)=\mu_A(x)-\nu_A(x)\geq 0$ and is a negative alternative for $d_A(x)=\mu_A(x)-\nu_A(x)<0$.

From Theorem 1, the property of the relation between preference and certain information differences is followed with respect to the alternative $d_A(x)$.

Corollary 1. For an IFS A, the preference score function $S_P\big(A(x)\big)$ strictly increases with respect to $d_A(x)$. By the chain rule and the partial derivative of

$$\frac{\partial\mu_A(x)}{\partial d_A(x)}=1 \text{ and } \frac{\partial\nu_A(x)}{\partial d_A(x)}=-1, \text{ then it is clear } \frac{\partial\phi\big(A(x)\big)}{\partial d_A(x)}>0.$$

With Definition 7, we obtain the relation of the hesitation degree $\pi_A(x)$ and preference score $S_P(A)$ as follows.

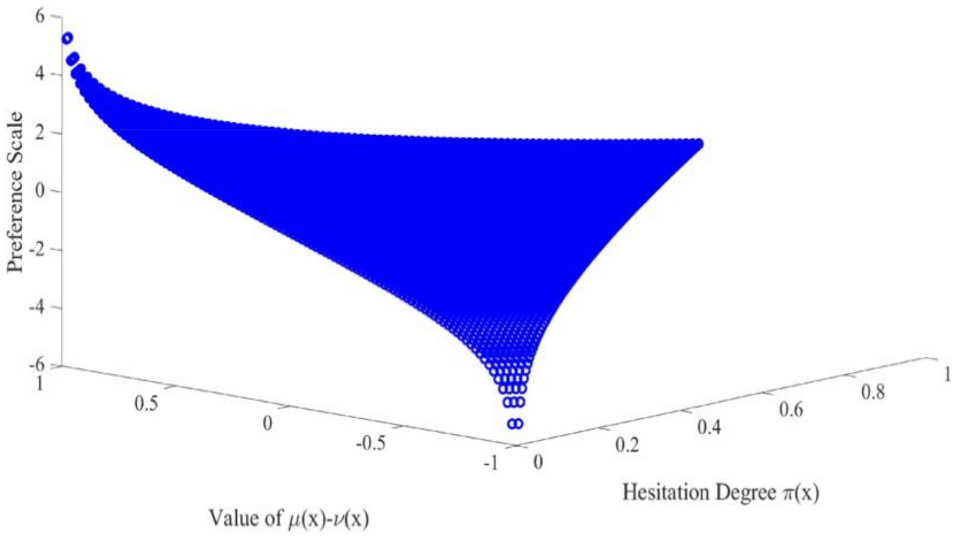

Fig. 4. Preference Score with Hesitation Degree and the Value of $\mu(x) - \nu(x)$

Corollary 2. If an IFS A is positive or negative alternative, then the preference score $S_P(A(x))$ strictly decreases or increase with respect to $\pi_A(x)$, respectively.

The Corollary 2 is clearly verified by applying partial derivative of $\phi(A)$ to $\pi_A(x)$ with the monotonicity of $acosh$. Visualization of Corollary 2 is illustrated in Fig.4. It indicates that the preference score $S_P(A)$ strictly decreases when the membership degree is bigger than the non-membership degree, and vice versa. Decreasing hesitation degree $\pi_A(x)$ makes the positive alternative better and the negative alternative worse. It implies that the hesitation degree $\pi_A(x)$ affects the significance of the preference score.

3.2.2. Preference score function for multi-criteria vector form and its property

Now, we define an operator for vectors to define the function for the scores based on the multicriteria vector form of IFSs.

Definition 8. For two arbitrary vectors $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$ with the same number of elements, a new operator $\odot$ is defined as:

$$A \odot B = (a_1 b_1, a_2 b_2, \dots, a_n b_n) \tag{17}$$

The operator provides the calculation of scores for the multicriteria vector form of IFSs containing weights for different criteria. Based on Definitions 2 and 5, the function of the novel preference score function for the multicriteria vector is defined in the following definition.

Definition 9. For IFS $A(x_i) = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle | x_i \in X\}$ with multi-criteria decision problem, the preference score $MS_P(A(x_i))$ is defined as:

$$MS_P(A(x_i)) = \begin{cases} acosh\left(1 + \delta\big(M_A(x), N_A(x)\big)\right), & \text{for } \|M_A(x)\| - \|N_A(x)\| \geq 0 \\ -acosh\left(1 + \delta\big(M_A(x), N_A(x)\big)\right), & \text{for } \|M_A(x)\| - \|N_A(x)\| < 0 \end{cases} \tag{18}$$

where $acosh$ denotes the arc hyperbolic cosine function, $\delta\big(M_A(x_i), N_A(x_i)\big)$ denotes an isometric invariant and expressed as:

$$\delta\big(M_A(x),N_A(x)\big)=\frac{\|M_A(x)-N_A(x)\|}{2\left(1-\frac{\|M_A(x)\|}{n}\right)^2\left(1-\frac{\|N_A(x)\|}{n}\right)^2} \tag{19}$$

where $\|\cdot\|$ denotes the l_2 norm, and $n>2$ is the number of criteria.

For the case of multicriteria decision-making problems with weights, we can change the isometric invariant as the following equation:

$$\delta\big(M_A(x),N_A(x)\big)=\frac{\|\omega\odot(M_A(x)-N_A(x))\,\|}{2(1-\|\omega\odot M_A(x)\|)^2(1-\|\omega\odot N_A(x)\|)^{\,2}} \tag{20}$$

where ω denotes the vector of weights for different criteria, and it is expressed as:

$$\omega=(\omega_1,\omega_2,\dots,\omega_n)$$

where $n>2$ is the number of criteria, $\omega_i\ \in\ \left[1,\frac{1}{n}\right]$, and $\sum_{i=1}^{n}\omega_i=\frac{1}{n}$.

The properties of Definition 9 are similar to those of Definition 6. The independent elements can be changed from $\mu_A(x)$ and $\nu_A(x)$ in (14) to $\|\,M_A(x)\|$ and $\|N_A(x)\|$ in (18), respectively. Then, the conclusions in Theorem 1 and Corollaries 1 and 2 can also be drawn for the preference score function based on the multicriteria vector form of IFSs.

4. Illustrative Examples

In this section, four illustrative examples are presented to demonstrate the utilization of the novel score function in decision-making problems. The examples also illustrate the comparative analysis with the existing score functions. Example 1 shows that the proposed score function S_P overcomes the situation that cannot be addressed in the previous preference functions [1,3,8,10]. Examples 2 and 3 illustrate a multicriteria decision-making problem. The proposed score function S_P provides a more significant result. In Example 4, actual data are applied, and the decision results are compared with the existing result [3].

Example 1. Table 1 shows two alternatives and four criteria decision-making problems in the existing research in [3]. There are two alternatives for each criterion, and we must choose one for the consideration of four criteria. The comparison with the existing score functions [1, 3, 8, 10] and the proposed S_P are illustrated. The scores of each score function are illustrated in Table 2. In Table 2, we denote A_{ij} as the i^{th} alternative in the j^{th} criteria, where $i=1,2$ and $j=1,2,3,4$.

Table 1
Two alternatives and four criteria

Alternative	Criteria			
	1	2	3	4
A_1	$\langle 0.8,0.2\rangle$	$\langle 0,0.1\rangle$	$\langle 0.7,0.3\rangle$	$\langle 0.67,0.18\rangle$
A_2	$\langle 0.7,0.1\rangle$	$\langle 0,0.8\rangle$	$\langle 0.85,0.15\rangle$	$\langle 0.68,0.25\rangle$

Table 2
Preference Scores

	Preference Scores of Example 1							
	A_{11}	A_{21}	A_{12}	A_{22}	A_{13}	A_{23}	A_{14}	A_{24}
[1]	0.6	0.6	- 0.1	- 0.8	0.4	0.7	0.49	0.43
[8]	0.8	0.84	0	0	0.4	0.85	0.77	0.73
[10]	N/A	0.97	- 1.1	- 1.2	N/A	N/A	0.77	0.59
[3]	1.8	1.6	0.48	0.34	1.5	2.0	1.44	1.44
S_P	1.8	1.5	- 0.15	- 2.1	1.2	2.3	1.24	1.17

where N/A denotes the preference score cannot be obtained due to the calculation of cross entropy of IFSs.

For a better understanding of the proposed method, we take the first criterion of alternative A_1 as an example. Since the IFS of A_1 is $\langle 0.8,0.2\rangle$ under criterion 1, the IFS in vector form of A_1 is $\langle(0.8,0.8),(0.2,0.2)\rangle$ according to Eq. (12). The preference score under criterion 1 is $S_P = arcosh\left(1+\frac{\|0.8-0.2\|}{2\times 0.8\times 0.2}\right) = 1.8$ by Eq. (15) and (16).

Table 2 shows the failures of previous approaches; the first criteria in [1]; the second criteria in [8]; the third in [10]; and the fourth in [3]. These were due to not analyzing the hesitation degree in the first criteria [1], ignorance of the non-membership degree in the second criteria [8], failure to address the case of $\pi(x) = 0$ in the third criterion [10], and insufficient precision in the last criteria [3]. Finally, the proposed score function S_P overcomes the shortcomings of the existing research.

From the fourth criterion in Table 1, there is a slight change in the membership and non-membership degrees in the IFSs, 0.01 in membership degree and 0.07 in non-membership degree. S_W had the highest variation in the result of 0.18 [10]. S_P had the second highest variation, with a result of 0.07. S_C and S_L obtained the third and fourth highest variation at 0.06 and 0.04, respectively [1, 5]. However, S_G cannot discriminate the preference due to insufficient precision [3].

Examining all of these score functions, we can see that score functions can solve various situations when mapping membership and non-membership to higher dimensional spaces, such as S_G and S_P. Since the rate of change of S_P follows an exponential growth [26], S_P can still perform well with some slight changes.

Example 2. Example 1 shows the advantages of the proposed score function S_P related to the vector form of IFSs defined by Definition 4. However, S_P may not effectively handle the decision-making problem because the denominator equals 0 in certain situations, e.g., the membership degree $\mu(x) = 1$ in one of the criteria. The multicriteria vector form allows IFSs to be more flexible in all criteria. Therefore, this example generated two alternatives, A_1 and A_2, with four criteria c_1 to c_4. Table 3 displays the details of IFSs for each criterion of alternatives.

Table 3
Two alternatives with four criteria

Alternatives	Criteria			
	c_1	c_2	c_3	c_4
A_1	$\langle 1,0\rangle$	$\langle 0.76,0.23\rangle$	$\langle 0.35,0.56\rangle$	$\langle 0.86,0.1\rangle$
A_2	$\langle 0.75,0.2\rangle$	$\langle 1,0\rangle$	$\langle 0.23,0.35\rangle$	$\langle 0.83,0.12\rangle$

Here, the weights of each criterion are considered equally in the decision-making problem. We can calculate the summation of the scores for each alternative and show the preference according to four criteria. The MS_P of each alternative is also calculated. Table 4 illustrates the results of the calculation.

Table 4
Preference Score of Example 2

Method	Score				
	c_1	c_2	c_3	c_4	Total
$SP(A1)$	N/A	1.078	-0.493	1.643	N/A
$SP(A2)$	1.114	N/A	-0.374	1.483	N/A
$MSP(A1)$	-	-	-	-	1.493
$MSP(A2)$	-	-	-	-	1.433

Where **N/A** denotes that the score cannot be obtained due to the 0 denominator.

The preference score can be calculated by Eq. (18) even if the membership degree is one. The difference between $MS_P(A_1)$ and $MS_P(A_2)$ is $MS_P(A_1) - MS_P(A_2) = 0.06$. It indicates $A_2 \prec A_1$. If we only consider the criteria that can be calculated by the score functions, the average scores of S_P for alternatives A_1 and A_2 are $\bar{S}_P(A_1) = 0.803$ and $\bar{S}_P(A_2) = 0.743$. The difference in scores between A_1 and A_2 is 0.06. The conclusion of the decision-making problem is the same as that in MS_P. Nonetheless, there is a lack of justification for simply ignoring the case where the denominator is zero.

Example 3. To perform sensitivity analysis and comparative analysis of the proposed method, we used a numerical example in [6], Table 5. This example includes four alternatives denoted as A_1, A_2, A_3, and A_4, and the IFSs are generated under five criteria as C_1, C_2, C_3, C_4, and C_5. The details of each IFS are shown below. The weight of the criteria is $w = (0.23,0.2,0.2,0.125,0.22)^T$.

Table 5
Details of IFSs in Example 3

Alternative	Criteria				
	C_1	C_2	C_3	C_4	C_5
A_1	$\langle 0.23,0.587\rangle$	$\langle 0.61,0.2\rangle$	$\langle 0.192,0.63\rangle$	$\langle 0.22,0.75\rangle$	$\langle 0.196,0.62\rangle$
A_2	$\langle 0.26,0.554\rangle$	$\langle 0.2,0.61\rangle$	$\langle 0.63,0.192\rangle$	$\langle 0.094,0.875\rangle$	$\langle 0.62,0.196\rangle$
A_3	$\langle 0.62,0.197\rangle$	$\langle 0.61,0.2\rangle$	$\langle 0.259,0.56\rangle$	$\langle 0.31,0.66\rangle$	$\langle 0.227,0.59\rangle$
A_4	$\langle 0.197,0.62\rangle$	$\langle 0.36,0.454\rangle$	$\langle 0.337,0.484\rangle$	$\langle 0.15,0.82\rangle$	$\langle 0.332,0.5\rangle$

After calculation by the score functions, we summarize the scores of each alternative in Table 6.

Table 6
Preference Score of Example 3

Method	Score				
	A_1	A_2	A_3	A_4	Rank
[1]	-0.247	-0.066	-0.045	-0.268	$A_4 \prec A_1 \prec A_2 \prec A_3$
[8]	0.332	0.441	0.474	0.321	$A_4 \prec A_1 \prec A_2 \prec A_3$
[10]	-0.259	-0.066	-0.002	-0.278	$A_4 \prec A_1 \prec A_2 \prec A_3$
[3]	0.691	0.864	0.896	0.643	$A_4 \prec A_1 \prec A_2 \prec A_3$
S_P	-0.679	-0.224	-0.080	-0.873	$A_4 \prec A_1 \prec A_2 \prec A_3$

All methods obtained the same rank of $A_4 \prec A_1 \prec A_2 \prec A_3$. This implies that the proposed method can make the same decision under the same conditions as other methods. The advantage of the proposed method is that the scores of different alternatives are more discriminative than the others.

Once we changed the weight of criteria as $w_1' = (0.3,0.3,0.2,0.1,0.1)^T$ and $w_2' = (0.125,0.125,0.2,0.3,0.25)^T$, we can obtain the rank of alternatives as $A_4 \prec A_2 \prec A_1 \prec A_3$, and $A_4 \prec A_1 \prec A_3 \prec A_2$. These results show that the proposed method can be influenced by different considerations from decision-makers.

Example 4. We use large-scale rooftop photovoltaic (LSRPV) projects as an example for the actual data analysis. Researchers have generated IFSs from real data as criteria for different alternatives [3].

In this case study, ten LSRPV projects are denoted as alternatives as A_1 to A_{10}. A_1 to A_{10} are located in different areas of China. The details of these ten alternatives are displayed in Table 7.

Table 7
Location of Alternatives

	Alternatives				
	A_1	A_2	A_3	A_4	A_5
City	Guangzhou	Huangzhou	Hefei	Jinan	Nanchang
Province	Guangdong	Zhejiang	Anhui	Shandong	Jiangxi
	Alternatives				
	A_6	A_7	A_8	A_9	A_{10}
City	Ningbo	Taizhou	Lishui	Foshan	Dongguan
Province	Zhejiang	Zhejiang	Zhejiang	Guangdong	Guangdong

There are three criterion types, with fifteen criteria in total constructed in [3]. The first type includes six criteria noted as C_{11}, C_{12}, C_{13}, C_{14}, C_{22}, and C_{23}. These criteria correspondingly indicate the initial investment cost, operation and maintenance costs, annual capital income, payback period, pollutant emission reduction benefits, and energy-saving benefits. The second type consists of five criteria, including C_{21}, C_{31}, C_{32}, C_{33}, and C_{45}. This type corresponds to light pollution, impact on the local economy, public support, policy support, and extreme weather. The last type includes C_{41}, C_{42}, C_{43}, and C_{44}. The third type includes annual sunshine hours, annual total solar radiation, average temperature, and available rooftop area.

The IFSs of different alternatives, which are generated by these three types of criteria, are displayed in Tables 8, 9, and 10.

Table 8
IFSs for the First Type Criteria [3]

Alt.	C_{11}	C_{12}	C_{13}	C_{14}	C_{22}	C_{23}
A_1	⟨0.33,0.65⟩	⟨0.36,0.64⟩	⟨0.27,0.73⟩	⟨0.30,0.69⟩	⟨0.25,0.73⟩	⟨0.25,0.74⟩
A_2	⟨0.28,0.71⟩	⟨0.28,0.72⟩	⟨0.34,0.66⟩	⟨0.33,0.67⟩	⟨0.32,0.67⟩	⟨0.32,0.66⟩
A_3	⟨0.26,0.73⟩	⟨0.25,0.75⟩	⟨0.36,0.63⟩	⟨0.33,0.66⟩	⟨0.35,0.63⟩	⟨0.35,0.63⟩
A_4	⟨0.40,0.58⟩	⟨0.38,0.61⟩	⟨0.24,0.76⟩	⟨0.33,0.66⟩	⟨0.26,0.72⟩	⟨0.26,0.72⟩
A_5	⟨0.41,0.58⟩	⟨0.31,0.68⟩	⟨0.29,0.70⟩	⟨0.30,0.69⟩	⟨0.26,0.71⟩	⟨0.27,0.72⟩
A_6	⟨0.41,0.58⟩	⟨0.36,0.62⟩	⟨0.23,0.77⟩	⟨0.31,0.67⟩	⟨0.23,0.74⟩	⟨0.23,0.74⟩
A_7	⟨0.30,0.69⟩	⟨0.33,0.66⟩	⟨0.31,0.68⟩	⟨0.32,0.67⟩	⟨0.31,0.67⟩	⟨0.31,0.67⟩
A_8	⟨0.26,0.73⟩	⟨0.26,0.73⟩	⟨0.35,0.64⟩	⟨0.31,0.68⟩	⟨0.36,0.60⟩	⟨0.36,0.60⟩
A_9	⟨0.23,0.76⟩	⟨0.25,0.74⟩	⟨0.37,0.62⟩	⟨0.24,0.74⟩	⟨0.35,0.62⟩	⟨0.35,0.62⟩
A_{10}	⟨0.28,0.71⟩	⟨0.30,0.69⟩	⟨0.34,0.65⟩	⟨0.32,0.66⟩	⟨0.32,0.65⟩	⟨0.32,0.65⟩

Table 9
IFSs of the Second Type Criteria [3]

Alt.	C_{21}	C_{31}	C_{32}	C_{33}	C_{45}
A_1	⟨0.54,0.35⟩	⟨0.54,0.34⟩	⟨0.79,0.14⟩	⟨0.72,0.20⟩	⟨0.51,0.38⟩
A_2	⟨0.65,0.25⟩	⟨0.67,0.23⟩	⟨0.77,0.16⟩	⟨0.54,0.34⟩	⟨0.57,0.32⟩
A_3	⟨0.73,0.18⟩	⟨0.73,0.18⟩	⟨0.73,0.18⟩	⟨0.75,0.17⟩	⟨0.72,0.20⟩
A_4	⟨0.56,0.32⟩	⟨0.61,0.28⟩	⟨0.61,0.28⟩	⟨0.73,0.18⟩	⟨0.58,0.31⟩
A_5	⟨0.67,0.23⟩	⟨0.65,0.25⟩	⟨0.78,0.15⟩	⟨0.70,0.21⟩	⟨0.65,0.25⟩
A_6	⟨0.79,0.14⟩	⟨0.49,0.38⟩	⟨0.75,0.17⟩	⟨0.75,0.17⟩	⟨0.40,0.47⟩
A_7	⟨0.72,0.20⟩	⟨0.72,0.20⟩	⟨0.76,0.16⟩	⟨0.70,0.21⟩	⟨0.58,0.30⟩
A_8	⟨0.52,0.37⟩	⟨0.54,0.34⟩	⟨0.78,0.15⟩	⟨0.58,0.30⟩	⟨0.58,0.30⟩
A_9	⟨0.58,0.31⟩	⟨0.62,0.27⟩	⟨0.72,0.20⟩	⟨0.78,0.15⟩	⟨0.45,0.44⟩
A_{10}	⟨0.65,0.25⟩	⟨0.72,0.20⟩	⟨0.54,0.34⟩	⟨0.75,0.17⟩	⟨0.51,0.38⟩

Table 10
IFSs of the Third Type Criteria [3]

Alt.	C_{41}	C_{42}	C_{43}	C_{44}
A_1	⟨0.27,0.73⟩	⟨0.30,0.70⟩	⟨0.26,0.74⟩	⟨0.28,0.72⟩
A_2	⟨0.28,0.72⟩	⟨0.32,0.68⟩	⟨0.35,0.65⟩	⟨0.33,0.67⟩
A_3	⟨0.31,0.69⟩	⟨0.32,0.68⟩	⟨0.35,0.65⟩	⟨0.37,0.63⟩
A_4	⟨0.41,0.59⟩	⟨0.37,0.63⟩	⟨0.43,0.57⟩	⟨0.23,0.77⟩
A_5	⟨0.34,0.66⟩	⟨0.28,0.72⟩	⟨0.32,0.68⟩	⟨0.32,0.68⟩
A_6	⟨0.31,0.69⟩	⟨0.32,0.68⟩	⟨0.31,0.69⟩	⟨0.23,0.77⟩
A_7	⟨0.32,0.68⟩	⟨0.32,0.68⟩	⟨0.30,0.70⟩	⟨0.31,0.69⟩
A_8	⟨0.26,0.74⟩	⟨0.30,0.70⟩	⟨0.29,0.71⟩	⟨0.27,0.63⟩
A_9	⟨0.32,0.68⟩	⟨0.32,0.68⟩	⟨0.26,0.74⟩	⟨0.37,0.63⟩
A_{10}	⟨0.32,0.68⟩	⟨0.33,0.67⟩	⟨0.26,0.74⟩	⟨0.31,0.69⟩

After calculation by the proposed score function, we can compare the scores of each alternative with previous score functions. We assume that the weight of each criterion is the same; then, the scores obtained by different score functions are shown in Table 11.

Table 11
Scores of Alternatives

Method	Alternatives									
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
[3]	0.88	0.91	1.01	0.91	0.95	0.91	0.96	0.88	0.90	0.90
S_P	- 0.52	- 0.43	- 0.30	- 0.40	- 0.38	- 0.53	- 0.37	- 0.47	- 0.47	- 0.44

From the scores in Table 9, the conclusion can be drawn as the ranking of the alternatives as $A_6 \prec A_1 \prec A_8 \prec A_9 \prec A_{10} \prec A_2 \prec A_4 \prec A_5 \prec A_7 \prec A_3$ by S_P. The ranking of alternatives by S_G is $A_1 \prec A_8 \prec A_9 \prec A_{10} \prec A_2 \prec A_4 \prec A_6 \prec A_5 \prec A_7 \prec A_3$. Here is a difference in the ranking order. However, we can still notice that the top three alternatives with preference are $A_5 \prec A_7 \prec A_3$ by S_P and S_G.

Even though the order of alternatives is different between the proposed method and the previous method, we can analyze the correlation between these two methods by Spearman's rank correlation coefficient [22]. For any two datasets a and b including the same number of alternatives, Spearman's rank correlation coefficient $\rho(a,b)$ between the two datasets can be expressed as:

$$\rho = 1 - \frac{\frac{1}{n}\sum_{i=1}^{n}\left(R(a_i)-\overline{R(a)}\right)\left(R(b_i)-\overline{R(b)}\right)}{\sqrt{\left(\frac{1}{n}\sum_{i=1}^{n}\left(R(a_i)-\overline{R(a)}\right)^2\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left(R(b_i)-\overline{R(b)}\right)^2\right)}} \quad (21)$$

where n is the number of alternatives and $R(\cdot)$ and $\overline{R(\cdot)}$ denote the rank of each alternative and the average rank for the dataset, respectively.

Eq. (21) can be simplified as follows:

$$\rho = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} \quad (22)$$

where d_i is the difference of the rank between corresponding alternatives.

Table 12
Ranks of Alternatives

Method	Alternatives									
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
[3]	9.5	5	1	5	3	5	2	9.5	7.5	7.5
S_P	9	5	1	4	3	10	2	7.5	7.5	6

Table 12 presents the rank of each alternative. We find that the alternatives A_1 and A_8, A_2, A_4 and A_6, and A_9 and A_{10} have the same rank by S_G. By the proposed method, only A_8 and A_9 have the same rank.

After calculation, the Spearman's correlation coefficient between S_P and S_G is $\rho(S_P, S_G) = 0.803$, and the p value $p = 0.0056$. This result shows that there is a strong positive correlation between S_G and the proposed method S_P.

5. Discussions

Realizing the use of the preference function with the score function is fundamental to solving the decision-making problem, and more concrete and differentiative measures are needed. Specifically, IFS data include hesitation when they constitute the total information together with $\mu(x)$ and $\nu(x)$. A graphical representation of $\pi(x)$ is illustrated by Figures 1 and 2, which help communicate the data behaviours. With the Poincaré metric, the score function designed with $\mu(x)$ and $\nu(x)$ showed strict monotonicity characteristics. Its characteristics are summarized by Theorem 1 and proven, and strict decreases and increases in the score function $S_P(A)$ with respect to $\mu(x)$ and $\nu(x)$ are also summarized in the subsequent corollary. Decreasing the hesitation degree $\pi(x)$ makes the positive alternative better and the negative alternative worse. This implies that the hesitation degree $\pi(x)$ affects the significance of the score function. For more general application, the score function is extended to the multicriteria decision problem. Definition 5 considers the multicriteria IFSs as two parts: $\mu_A(x)$ and $\nu_A(x)$. The membership degrees of different criteria are considered as the independent variable. The score function structures of the single criterion and multicriteria are similar; Definitions 6 and 9. It is notified only from $\mu_A(x)$ and $\nu_A(x)$ to $\|M_A(x)\|$ and $\|N_A(x)\|$ in Eq. (18), respectively. The score function of the multicriteria vector form of IFSs is summarized in Theorem 1 and Corollaries 1 and 2.

The proposed score function is applied to the existing examples to verify its usefulness. By the numerical examples, the proposed method shows its effectiveness in solving the conditions that cannot make clear decisions by existing methods. Meanwhile, the essence of sensitivity and comparative analysis is also illustrated by a multicriteria decision-making example. Under the same conditions, the same result can be drawn by the proposed method and the existing methods. Meanwhile, the influence of decision-makers on the proposed method is also illustrated by the examples. With the help of the exponential changing rate from the Poincaré metric, the scores obtained by the proposed method are more discriminative when the changes are slight in IFSs.

The obtained research provides an effective measure to discriminate attributes with strict monotonicity characteristics. Even IFSs include comprehensive information; support, descent, absence, and $\pi(x)$ considerations show clear differentiation in decision-making.

6. Conclusions

We have proposed a new score function with the help of the Poincaré metric for decision-making with IFSs, which is based on a pair of independent two-dimensional vectors composed of the membership and non-membership degrees of an IFS; $\mu(x)$ and $\nu(x)$. In this regard, we propose strict monotonic properties in Theorem 1, Corollary 1, and Corollary 2; in particular, the score function dependency on the difference between membership and non-membership degrees is emphasized in Corollary 1. Compared with the existing research, the proposed score function showed clear decision results with the help of strict monotonic properties.

Through illustrative examples, we have also demonstrated the advantages of the proposed score function over existing ones; the proposed score function overcomes the shortcomings of the previous research on the single-criteria decision-making problem. By mapping the membership and non-membership degrees to a high-dimensional space, the proposed score function can achieve equivalent scores as the previous methods. Due to the strict monotonicity of the proposed method, it is possible to obtain a more discriminative result when the membership and non-membership degrees are similar in different alternatives. Additionally, it achieves more flexible decision results in multicriteria problems. Finally, actual data were applied and showed consistency in high preference results. Due to the exponential changing rate, the score function based on the Poincaré metric outperformed previous methods when the changes in membership and non-membership degrees were slight.

Research outputs make it possible to extend actual data decision making with the obtained research not only single criterion but also multi-criteria. Furthermore, independent vector processing with Poincaré metric helps to treat to general data tasking. However, data fuzzification is the challenge; specifically, to IFSs. It was carried out through subjective ways – personally. Data fuzzification to the objective method should be followed for more application.

Author Contributions

Conceptualization, S.L; methodology, S.L, Y.Y; software, Y.Y; validation, S.L, K.K; formal analysis, S.L, X.H, Y.Y; investigation, Y.Y; resources, Y.Y; data curation, Y.Y; writing—original draft preparation, S.L, Y.Y; writing—review and editing, H.Z, X.H; visualization, K.K, Y.Y; supervision, W.P, S.L; project administration, S.L; funding acquisition, S.L; All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing financial interests and personal relationships that could have appeared to influence the research in this paper.

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