Decision Making: Applications in Management and Engineering
Vol. 6, Issue 1, 2023, pp. 474-501.
ISSN: 2560-6018
eISSN: 2620-0104
DOI: https://doi.org/10.31181/dmame24022023b

COMPLEX FERMATEAN NEUTROSOPHIC GRAPH AND APPLICATION TO DECISION MAKING

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Received: 27 November 2022;
Accepted: 20 February 2023;
Available online: 24 February 2023.

Original scientific paper

Abstract: *A new growing area of the neutrosophic set theory called complex neutrosophic sets (CNS) provides useful tools for dealing with uncertainty in complex valued physical variables that are observed in the actual world. A CNS takes values for the truth, indeterminacy and falsity membership functions in the complex plane's unit circle. In this research, a novel concept of complex fermatean neutrosophic graph (CFNG) is established and various basic graphical ideas such as the order, size, degree and total degree of a vertex of it are introduced. Also, set theoretical operations such as complement, union, join, ring-sum and cartesian product of CFNG are studied. Further, the concept of a regular graph under a complex fermatean neutrosophic environment is introduced. Finally, we make use of the proposed CFNG in solving a multi-criteria decision-making problem in which the graphical structure of attributes is uncertain. This study also demonstrates the application of a CFNG in the educational system to evaluate a lecturer's research productivity.*

Key words: *Complex neutrosophic sets, Complex fermatean Neutrosophic sets, Neutrosophic graphs, Complex Fermatean neutrosophic graph.*

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1. Introduction

The idea of fuzzy sets (FS) was initially proposed by (Zadeh, 1965). It is very good at coping with the uncertainty that exists in a real-world decision-making. FS considers an element's membership degree (α) alone and does not allow us to independently select non-membership degree (β). In this sense, fuzzy sets are still somewhat classical. However, this fact may be considered as a disadvantage. Thus, (Atanassov, 1986) proposed intuitionistic fuzzy sets [IFS] which facilitate the decision maker (DM) to assign α and β independently with the condition $0 \le \alpha + \beta \le 1$. By relaxing the restriction of IFS to $0 \le \alpha^2 + \beta^2 \le 1$, Yager (2013, 2014) proposed Pythagorean fuzzy sets [PFS]. Senapati and Yager (2019, 2019a) proposed Fermatean fuzzy set (FFS) by further relaxing the restriction of PFS $0 \le \alpha^3 + \beta^3 \le 1$ to handle a wider range of uncertainty. A comparison between the space of FS, IFS, PFS, FFS, and PS is shown in Table 1.

Table 1. Comparison between FS, IFS, PFS, FFS, and PS

Extension	Condition	Example	Explanation
FS	$0 \le \alpha \le 1$	$A = \{(a,0.1),(b,0.5)\}$	Doesn't consider β
IFS	$0 \le \alpha + \beta \le 1$	$A = \{(a,0.3,0.5),(b,0.6,0.3)\}$ $B = \{(a,0.5,0.7),(b,0.7,0.4)\}$	Condition for IFS satisfies for A but fails for B as $\alpha + \beta > 1$
PFS	$0 \le \alpha^2 + \beta^2 \le 1$	$B = \{(a,0.4,0.8),(b,0.5,0.6)\}$ $C = \{(a,0.7,0.8),(b,0.9,0.6)\}$	Condition for PFS satisfies for B but fails for C as $\alpha^2 + \beta^2 > 1$
FFS	$0 \le \alpha^3 + \beta^3 \le 1$	$C = \{(a,0.7,0.8),(b,0.9,0.6)\}$	Condition for FFS satisfies for C
NS	$0 \le \alpha + \beta + \gamma \le 3$	$D = \{(a,0.7,0.6,0.9),(b,0.6,0.8,0.7)\}$	Includes γ

One can add that there also other solutions of this kind. For example, Ibrahim et al. (2021) introduced (3,2)-fuzzy sets while Shanmathi and Nirmala (2022) studied (4,2)-fuzzy sets. In both cases, appropriate topological spaces were studied too. As for the neutrosophic sets (NS), they were initiated by (Smarandache (1999). They incorporate the indeterminacy membership degree (γ) along with the truth membership degree (α) and falsity membership degree (β). Details will be described later in the paper. Another aspect of our work is related to complex numbers and functions. Many physical quantities in practical problems such as wave function in quantum mechanics are complex-valued. Buckley (1989) first initiated the formal definition of complex fuzzy numbers (CFN) and their types. Later on, Ramot et al. (2002) presented the concept of complex fuzzy sets (CFS) by extending

FS in the real field to the complex field. The membership degree of CFS is of the form $\alpha.e^{i\omega}$ where $i=\sqrt{-1}$ whose amplitude term α lies in [0,1] and phase term ω lies in $[0,2\pi]$. Many researchers have focused their attention on CFS over the past few years. Yazdanbakhsh and Dick (2018) reviewed CFS and highlighted potential future research topics in it. Many researchers investigated and worked on CFS and its extensions. Alkouri and Salleh (2012) generalized the concept of CFS to IFS and proposed a complex intuitionistic fuzzy set (CIFS) whose membership and non-membership functions are complex valued. Ullah et al. (2019) presented the concept of complex Pythagorean fuzzy set (CPyFS). Chinnadurai et al. (2021) presented complex Fermatean fuzzy set (CFFS) in rectangular coordinates and its applications in decision-making problems. Ali and Smarandache (2017) proposed complex neutrosophic sets (CNS). Extensions of CFS under various fuzzy environment are shown in Table 2.

Table 2. Complex fuzzy sets under various fuzzy environment

References	Extensions
(Ramot et al., 2002)	CFS
(Alkouri and Salleh, 2012)	CIFS
(Ullah et al., 2019).	CPyFS
(Chinnadurai et al., 2021)	CFFS
(Ali and Smarandache, 2017)	CNS

Our paper deals with graph theory too (but in a context of uncertainty). Therefore, we may say that fuzzy graphs (FG) are developed to model uncertainties in graphical network and their extensions are explored by many researchers (Saumya and Hegde, 2022). Thirunavukarasu et al. (2016) extended FG to complex fuzzy graphs (CFG) and proposed energy of CFG. Yaqoob and Akram (2018) studied about complex neutrosophic graphs (CNG). Also, Yaqoob et al. (2019) suggested the complex intuitionistic fuzzy graphs (CIFG) using some basic operations. Akram and Naz (2019) proposed complex Pythagorean fuzzy graphs (CPFG). Shoaib et al. (2022) introduced the concept of picture fuzzy graphs and their properties. Also, Shoaib et al. (2022a) initiated the notion of complex spherical fuzzy graph. Pythagorean neutrosophic fuzzy graphs (PFNG) was suggested by Ajay and Chellamani (2020). Antony and Jansi (2021) proposed the fermatean neutrosophic sets (FNS) and Broumi et al. (2022) explored the fermatean neutrosophic graphs (FNG) with their applications. Motivated by these works we initiated the concept, complex fermatean neutrosophic fuzzy sets (CFNS) and complex fermatean neutrosophic fuzzy graph (CFNG). Butt et al. (2022) established a complex dombi fuzzy graph and their properties. In the single valued neutrosophic environment, regular graphs are explored by Samina et al. (2017). Uncertain decision-making problems can be easily organized and modeled using CFNG theory.

An overview of the motivation for this paper is provided below:

1. A complex Fermatean neutrosophic set (CFNS) is capable of handling the situation well when faced with imprecise and intuitive knowledge in an uncertain decision-making process.
2. Due to the CFNS's phase term, there is no information loss, and are incredibly effective at making decisions and have a wider range of applications.

The primary contributions of this work are as follows:

1. First and foremost, we initiated the new idea of CFNGs.
2. We studied the properties of CFNGs.

3. We introduced the order, degree, size and total degree of CFNGs.
4. We introduced the operations of CFNG such as complement, union, join, ring sum and cartesian product of CFNG.
5. Also studied about the regularity of CFNG and its properties.
6. Discussed the application of CFNG in MCDM problems using the proposed arithmetic and geometric operators.

This paper is arranged as follows. In part 2, we presented the preliminary findings relating to CNG. Some important properties of CFNGs such as the order, size, primary operations, degree, total degree and regularity are introduced in section 3. In section 4, an application of CFNG in educational system is discussed. The conclusion and forthcoming work are discussed in section 5.

2. Preliminaries

This section provides an introduction to few fundamental definitions of CNS to further explain the innovative idea of CFNS.

2.1. Some important definitions

Definition 2.1 (Smarandache, 1999): A neutrosophic set (NS) S on a non-empty universe X is described by $\mathrm{T}_S, \mathrm{I}_S, \mathrm{F}_S : X \to]0^-, 0^+[$. That is

$$S = \{(\mathrm{u}, \mathrm{T}_S(u), \mathrm{I}_S(u), \mathrm{F}_S(u)) : u \in X\}$$

with $0^- \le \mathrm{T}_S(u) + \mathrm{I}_S(u) + \mathrm{F}_S(u) \le 3^+$. Here $\mathrm{T}_S(u), \mathrm{I}_S(u)$ and $\mathrm{F}_S(u)$ are truth, indeterminacy and falsity membership values respectively and are real standard or non-standard subsets of $]0^-, 0^+[$. It is worth to mention that the definition above is very general. In particular, it allows to combine neutrosophic sets with some ideas taken from the world of non-standard analysis. However, the vast majority of researchers limit their interest to single-valued neutrosophic sets. In this case T, I and F are just real numbers from [0, 1]. This approach is more practical. Also interval-valued neutrosophic sets are studied.

Definition 2.2 (Ajay and Chellamani, 2020): A Pythagorean neutrosophic set (PNS) S defined on X is represented as

$$S = \{(\mathrm{u}, \mathrm{T}_S(u), \mathrm{I}_S(u), \mathrm{F}_S(u)) : u \in X\}$$

with $0^- \le \mathrm{T}_S^2(u) + \mathrm{I}_S^2(u) + \mathrm{F}_S^2(u) \le 2^+$ and $0 \le \mathrm{T}_S^2(u) + \mathrm{F}_S^2(u) \le 1;\ 0 \le \mathrm{I}_S^2(u) \le 1$. Here $\mathrm{T}_S(u), \mathrm{F}_S(u)$ are dependent components and $\mathrm{I}_S(u)$ is an independent component.

Definition 2.3 (Antony and Jansi, 2021): A Fermatean neutrosophic set (FNS) S defined on X is represented as

$$S = \{(\mathrm{u}, \mathrm{T}_S(u), \mathrm{I}_S(u), \mathrm{F}_S(u)) : u \in X\}$$

with $0^- \le \mathrm{T}_S^3(u) + \mathrm{I}_S^3(u) + \mathrm{F}_S^3(u) \le 2^+$ and $0 \le \mathrm{T}_S^3(u) + \mathrm{F}_S^3(u) \le 1;\ 0 \le \mathrm{I}_S^3(u) \le 1$.

Definition 2.4 (Ali and Smarandache, 2017): A complex neutrosophic set (CNS) S is described on X is represented as

$$S=\left\{\left(\mathrm{u},\mathrm{T}_S\left(u\right),\mathrm{I}_S\left(u\right),\mathrm{F}_S\left(u\right)\right):u\in X\right\}$$

where $\mathrm{T}_S\left(u\right),\mathrm{I}_S\left(u\right),\mathrm{F}_S\left(u\right)$ are complex-valued functions and are of the form $\mathrm{T}_S\left(u\right)=r_S\left(u\right).e^{i\omega_S(\mathrm{u})}$; $\mathrm{F}_S\left(u\right)=k_S\left(u\right).e^{i\rho_S(\mathrm{u})}$; $\mathrm{I}_S\left(u\right)=t_S\left(u\right).e^{i\theta_S(\mathrm{u})}$; where $r_S\left(u\right),\omega_S(u),k_S\left(u\right),\rho_S(u),t_S\left(u\right),\theta_S(u)\in[0,1]$ such that $0^-\le r_S\left(u\right)+k_S\left(u\right)+t_S\left(u\right)\le 3^+$. The amplitude terms of CNS are $r_S\left(u\right),k_S\left(u\right),t_S\left(u\right)$ and the phase terms are $\omega_S(u),\rho_S(u),\theta_S(u)$. The triplet $\delta=\left(r.e^{i2\pi\omega_S},t.e^{i2\pi\theta_S},k.e^{i2\pi\rho_S}\right)$ is referred to as a complex Neutrosophic number.

Definition 2.5 (Yaqoob and Akram, 2018; Khan et al., 2022): A mapping $Q=\left(\mathrm{T}_Q,\mathrm{I}_Q,\mathrm{F}_Q\right):X\times X\to[0,1]$ is said to be a neutrosophic relation on X for $\mathrm{T}_Q\left(u,v\right),\mathrm{I}_Q\left(u,v\right),\mathrm{F}_Q\left(u,v\right)\in[0,1]$ where $u,v\in X$.

Definition 2.6 (Yaqoob and Akram, 2018): A neutrosophic graph (NG) on X is a pair $G=\left(P,Q\right)$ where P and Q are NS defined on vertex set V and edge set $E\subseteq V\times V$ respectively such that

$$\begin{cases}T_Q\left(u,v\right)\le\min\left\{T_P\left(u\right),T_P\left(v\right)\right\}\\ \mathrm{I}_Q\left(u,v\right)\le\min\left\{I_P\left(u\right),\mathrm{I}_P\left(v\right)\right\}\\ F_Q\left(u,v\right)\le\max\left\{F_P\left(u\right),\mathrm{F}_P\left(v\right)\right\}\end{cases}$$

Such that $0\le T_Q\left(u,v\right)+\mathrm{I}_Q\left(u,v\right)+F_Q\left(u,v\right)\le 3$ for all $u,v\in V$.

Definition 2.7 (Ajay and Chellamani, 2020): A Pythagorean neutrosophic graph (PyNG) on X is a pair $G=\left(P,Q\right)$ where P and Q are PyNS defined on vertex set V and edge set $E\subseteq V\times V$ respectively such that

$$\begin{cases}T_Q\left(u,v\right)\le\min\left\{T_P\left(u\right),T_P\left(v\right)\right\}\\ \mathrm{I}_Q\left(u,v\right)\le\min\left\{I_P\left(u\right),\mathrm{I}_P\left(v\right)\right\}\\ F_Q\left(u,v\right)\le\max\left\{F_P\left(u\right),\mathrm{F}_P\left(v\right)\right\}\end{cases}$$

Such that $0\le T_Q^2\left(u,v\right)+I_Q^2\left(u,v\right)+F_Q^2\left(u,v\right)\le 2$ for all $u,v\in V$.

Definition 2.8 (Antony and Jansi, 2021): A Fermatean neutrosophic graph (FNG) on X is a pair $G=\left(P,Q\right)$ where P and Q are FNS defined on vertex set V and edge set $E\subseteq V\times V$ respectively such that

$$\begin{cases}T_Q\left(u,v\right)\le\min\left\{T_P\left(u\right),T_P\left(v\right)\right\}\\ \mathrm{I}_Q\left(u,v\right)\le\min\left\{I_P\left(u\right),\mathrm{I}_P\left(v\right)\right\}\\ F_Q\left(u,v\right)\le\max\left\{F_P\left(u\right),\mathrm{F}_P\left(v\right)\right\}\end{cases}$$

Such that $0\le T_Q^3\left(u,v\right)+I_Q^3\left(u,v\right)+F_Q^3\left(u,v\right)\le 2$ for all $u,v\in V$.

Definition 2.9 (Yaqoob and Akram, 2018): A complex neutrosophic graph (CNG) on X is a pair $G=\left(P,Q\right)$ where P and Q are CNS defined on vertex set V and edge set $E\subseteq V\times V$ respectively such that

$$\begin{cases} r_Q(u,v).e^{i\omega_Q(u,v)} \leq \min\{r_P(u), r_P(v)\}.e^{i\min\{\omega_P(u),\omega_P(v)\}} \\ t_Q(u,v).e^{i\theta_Q(u,v)} \leq \min\{t_P(u), t_P(v)\}.e^{i\min\{\theta_P(u),\theta_P(v)\}} \\ k_Q(u,v).e^{i\rho_Q(u,v)} \leq \max\{k_P(u), k_P(v)\}.e^{i\min\{\rho_P(u),\rho_P(v)\}} \end{cases}$$

Where Q is complex neutrosophic relation on P.

3. Complex Fermatean neutrosophic graphs

This section explores the concept of the Complex Fermatean Neutrosophic Set (CFNS) and the Complex Fermatean Neutrosophic Graph (CFNG) as well as their fundamental operations.

3.1. Proposed CFNS and some important definitions

Definition 3.1: A complex Pythagorean neutrosophic set (CPNS) S on X is given by

$$S = \{(u, T_S(u), I_S(u), F_S(u)) : u \in X\}$$

where $T_S(u) = r_S(u).e^{i\omega_S(u)}$; $F_S(u) = k_S(u).e^{i\rho_S(u)}$; $I_S(u) = t_S(u).e^{i\theta_S(u)}$; with $0^- \leq r_s^2(u) + t_s^2(u) + k_s^2(u) \leq 2^+$ and $0^- \leq \omega_s^2(u) + \theta_s^2(u) + \rho_s^2(u) \leq 2\pi$.

Definition 3.2: A complex Fermatean neutrosophic set (CFNS) S on X is given by

$$S = \{(u, T_S(u), I_S(u), F_S(u)) : u \in X\}$$

with $0^- \leq r_s^3(u) + t_s^3(u) + k_s^3(u) \leq 2^+$ and $0^- \leq \omega_s^3(u) + \theta_s^3(u) + \rho_s^3(u) \leq 2\pi$. The triplet $\delta = (r \cdot e^{2\pi i\omega_S}, t \cdot e^{2\pi i\theta_S}, k \cdot e^{2\pi i\rho_S})$ is referred as a complex Fermatean Neutrosophic number (CFNN). A comparison of PNS, FNS, CNS and CFNS is made in Table 3.

Table 3. Comparison of PNS, FNS, CNS and CFNS

Extension	Condition	Example	Explanation
PNS	$0 \leq \alpha^2 + \beta^2 \leq 1$; $0 \leq \gamma^2 \leq 1$ *and* $0 \leq \alpha^2 + \beta^2 + \leq \gamma^2 \leq 2$	$A = \begin{Bmatrix} (a, 0.7, 0.6, 0.3), \\ (b.0.6, 0.5, 0.6) \end{Bmatrix}$ $B = \begin{Bmatrix} (a, 0.8, 0.8, 0.9), \\ (b, 0.9, 0.7, 0.9) \end{Bmatrix}$	Condition for PNS satisfies for A but fails for B as $\alpha^2 + \beta^2 > 1$
FNS	$0 \leq \alpha^3 + \beta^3 \leq 1$; $0 \leq \gamma^3 \leq 1$ *and* $0 \leq \alpha^3 + \beta^3 + \leq \gamma^3 \leq 2$	$B = \begin{Bmatrix} (a, 0.8, 0.8, 0.9), \\ (b, 0.9, 0.7, 0.9) \end{Bmatrix}$	Condition for FNS satisfies for A

Extension	Condition	Example	Explanation
CNS	$0 \le r+t+k \le 3$; $0 \le \omega+\theta+ \le \rho \le 2\pi$	$A=\left\{\begin{matrix}(a,\left\langle 0.1e^{i2\pi(0.3)},0.2e^{i2\pi(0.5)},0.7e^{i2\pi(0.2)}\right\rangle),\\(b,\left\langle 0.3e^{i2\pi(0.4)},0.2e^{i2\pi(0.2)},0.4e^{i2\pi(0.2)}\right\rangle)\end{matrix}\right\}$ $B=\left\{\begin{matrix}(a,\left\langle 0.8e^{i2\pi(0.7)},0.7e^{i2\pi(0.5)},0.7e^{i2\pi(0.2)}\right\rangle),\\(b,\left\langle 0.6e^{i2\pi(0.8)},0.6e^{i2\pi(0.8)},0.4e^{i2\pi(0.5)}\right\rangle)\end{matrix}\right\}$	Condition for CNS satisfies for A but fails for B as $r+t>1$
Proposed CFNS	$0 \le r^3+t^3+k^3 \le 3$; $0 \le \omega^3+\theta^3+ \le \rho^3 \le 2\pi$	$B=\left\{\begin{matrix}(a,\left\langle 0.8e^{i2\pi(0.7)},0.7e^{i2\pi(0.5)},0.7e^{i2\pi(0.2)}\right\rangle),\\(b,\left\langle 0.6e^{i2\pi(0.8)},0.6e^{i2\pi(0.8)},0.4e^{i2\pi(0.5)}\right\rangle)\end{matrix}\right\}$	Condition for CFNS satisfies for B

Definition 3.3: Let $\delta=\left(r\cdot e^{2\pi i\omega},t\cdot e^{2\pi i\theta},k\cdot e^{2\pi i\rho}\right),\delta_1=\left(r_1\cdot e^{2\pi i\omega_1},t_1\cdot e^{2\pi i\theta_1},k_1\cdot e^{2\pi i\rho_1}\right)$ and $\delta_2=\left(r_2\cdot e^{2\pi i\omega_2},t_2\cdot e^{2\pi i\theta_2},k_2\cdot e^{2\pi i\rho_2}\right)$ be three CFNNs, then

- $\delta_1 \subseteq \delta_2$ iff $r_1 \le r_2, t_1 \le t_2, k_1 \le k$ and $\omega_1 \le \omega_2,\ \theta_1 \le \theta_2, \rho_1 \le \rho_2$
- $\delta_1 = \delta_2$ iff $r_1 = r_2, t_1 = t_2, k_1 = k_2$ and $\omega_1 = \omega_2,\ \theta_1 = \theta_2, \rho_1 = \rho_2$
- $\overline{\delta}=\left(k\cdot e^{2\pi i\rho},t\cdot e^{2\pi i\theta},r\cdot e^{2\pi i\omega}\right)$

In the next definition we show how to perform operations of union and intersection on CFNNs.

Definition 3.4 Let $\delta_1=\left(r_1\cdot e^{2\pi i\omega_1},t_1\cdot e^{2\pi i\theta_1},k_1\cdot e^{2\pi i\rho_1}\right)$ and $\delta_2=\left(r_2\cdot e^{2\pi i\omega_2},t_2\cdot e^{2\pi i\theta_2},k_2\cdot e^{2\pi i\rho_2}\right)$ be two CFNNs, then

$$\delta_1 \cup \delta_2=\left\{x,\left(\max\left(r_1,r_2\right)\cdot e^{2\pi i\max(\omega_1,\omega_2)},\max\left(t_1,t_2\right)\cdot e^{2\pi i\max(\theta_1,\theta_2)},\max\left(r_1,r_2\right)\cdot e^{2\pi i\max(\rho_1,\rho_2)}\right)\right\}$$

$$\delta_1 \cap \delta_2=\left\{x,\left(\min\left(r_1,r_2\right)\cdot e^{2\pi i\min(\omega_1,\omega_2)},\min\left(t_1,t_2\right)\cdot e^{2\pi i\min(\theta_1,\theta_2)},\min\left(r_1,r_2\right)\cdot e^{2\pi i\min(\rho_1,\rho_2)}\right)\right\}$$

Definition 3.5 **:** A CFN relation in X is defined by a CFNS Q in $X\times X$ characterized by

$$Q=\left\{\left\langle\left(u,v\right),r_Q\left(u,v\right)\cdot e^{i\omega_Q(u,v)},t_Q\left(u,v\right)\cdot e^{i\theta_Q(u,v)},k_Q\left(u,v\right)\cdot e^{i\rho_Q(u,v)}\right\rangle\right\}$$

where $r_Q,t_Q,k_Q : X\times X\to\left[0,1\right]$ are defined such that $0\le r_Q^3\left(u,v\right)+t_Q^3\left(u,v\right)+k_Q^3\left(u,v\right)\le 2$ and $0\le\frac{\omega_Q^3\left(u,v\right)}{2\pi}+\frac{\theta_Q^3\left(u,v\right)}{2\pi}+\frac{\rho_Q^3\left(u,v\right)}{2\pi}\le 2$ where $\left(u,v\right)\in X\times X$ **.**

Now we shall connect our sets with some notions from graph theory.

Definition 3.6 **:** A complex fermatean neutrosophic graph (CFNG) defined on X is a pair $G=\left(P,\mathrm{Q}\right)$ where P is an CFNS on vertex set V and Q is a CFNS on edge set $E\subseteq V\times V$ such that

$$\begin{cases} r_Q(u,v)\cdot e^{i\omega_Q(u,v)} \le \min\{t_P(\mathrm{u}), t_P(\mathrm{v})\}\cdot e^{i\min\{\omega_P(\mathrm{u}),\omega_P(\mathrm{v})\}} \\ t_Q(u,v)\cdot e^{i\theta_Q(u,v)} \le \min\{r_P(\mathrm{u}), r_P(\mathrm{v})\}\cdot e^{i\min\{\theta_P(\mathrm{u}),\theta_P(\mathrm{v})\}} \\ t_Q(u,v)\cdot e^{i\rho_Q(u,v)} \le \min\{k_P(\mathrm{u}), k_P(\mathrm{v})\}\cdot e^{i\min\{\rho_P(\mathrm{u}),\rho_P(\mathrm{v})\}} \end{cases}$$

Where $r_P(u), k_P(u), t_P(u) \in [0,1], 0 \le r_Q^3(u,v), t_Q^3(u,v), k_Q^3(u,v) \le 2$ and $0 \le \frac{\omega_Q^3(u,v)}{2\pi} + \frac{\theta_Q^3(u,v)}{2\pi} + \frac{\rho_Q^3(u,v)}{2\pi} \le 2$ for all $(u,v) \in E$.

Example 1: Consider a CFNG $G=(P,Q)$ defined on $G=(V,E)$ represented in Figure 1. The vertex set P is a CFNS defined on V given in Table 4 and Q is a CFNS on edge set $E \subseteq V \times V$ given in Table 5.

Table 4. Vertices of CFNG G

Node	CFNS
v_1	$\langle 0.5e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.2)} \rangle$
v_2	$\langle 0.8e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.9)} \rangle$
v_3	$\langle 0.3e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.8)}, 0.4e^{i2\pi(0.5)} \rangle$

Table 5. Edges of CFNG G

Edge	CFFS
e_{12}	$\langle 0.3e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.2)} \rangle$
e_{13}	$\langle 0.3e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.5)} \rangle$
e_{23}	$\langle 0.2e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.4)} \rangle$

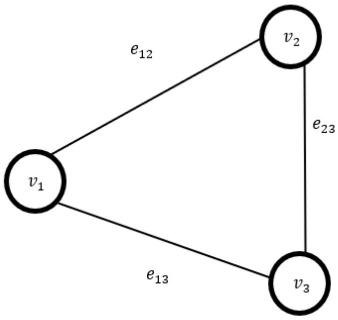

Figure 1. CFNG G

Definition 3.7*:* Consider a CFNG $G=(P,Q)$ whose vertex set is $P=\{\langle u, r_P(u)\cdot e^{i\omega_P(u)}, t_P(u)\cdot e^{i\theta_P(u)}, k_P(u)\cdot e^{i\rho_P(u)}\rangle \mid u \in V\}$ and edge set

$$Q=\{\langle u, r_Q(\mathrm{u,v})\cdot \mathrm{e}^{i\omega_Q(\mathrm{u,v})}, \mathrm{t}_Q(\mathrm{u,v})\cdot \mathrm{e}^{i\theta_Q(\mathrm{u,v})}, \mathrm{k}_Q(\mathrm{u,v})\cdot \mathrm{e}^{i\rho_Q(\mathrm{u,v})}\rangle \mid (u,v)\in E\}$$

then the order and size of a CFNG is defined by

$$O(G)=\left(\sum_{u\in V} r_P(u)\cdot e^{\sum_{u\in V}\omega_P(u)}, \sum_{u\in V} t_P(u)\cdot e^{\sum_{u\in V}\theta_P(u)}, \sum_{u\in V} k_P(u)\cdot e^{\sum_{u\in V}\rho_P(u)}\right)$$

$$S(G)=\left(\sum_{u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\in E}\omega_Q(u,v)}, \sum_{u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\in E}\theta_Q(u,v)}, \sum_{u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\in E}\rho_Q(u,v)}\right)$$

Example 2: The order and the size of the CFNG G represented in Figure 1 is given by $O(G)=\langle 1.6e^{i2\pi(1.8)}, 1.4e^{i2\pi(1.9)}, 1.7e^{i2\pi(1.6)}\rangle$ and $S(G)=\langle 0.8e^{i2\pi(1)}, 0.6e^{i2\pi(1.6)}, 1.1e^{i2\pi(0.9)}\rangle$ respectively.

3.2. Basic operations of complex fermatean fuzzy graphs

This section introduces some of the fundamental CFNG operations and explores some of its associated properties.

Definition 3.8*:* The complement of a CFNG $G=(P,Q)$ on an underlying graph $G=(V,E)$ is a CFNG $\overline{G}=(\overline{P},\overline{Q})$ defined by

- $\overline{r_P(u)\cdot e^{i\omega_Q(u)}} = r_P(u)\cdot e^{i\omega_Q(u)}, \overline{t_P(u)\cdot e^{i\theta_Q(u)}} = t_P(u)\cdot e^{i\theta_Q(u)}$ and $\overline{k_P(u)\cdot e^{i\rho_Q(u)}} = k_P(u)\cdot e^{i\rho_Q(u)}$.
- $$\overline{r_Q(u,v)\cdot e^{i\omega_Q(u,v)}} = \begin{cases} \min(r_P(u), r_P(v))\cdot e^{i\min(\omega_P(u),\omega_P(v))} & \text{if } r_Q(u,v)\cdot e^{i\omega_Q(u,v)}=0 \\ \min(r_P(u), r_P(v))\cdot e^{i\min(\omega_P(u),\omega_P(v))} - r_Q(u,v)\cdot e^{i\omega_Q(u,v)} & \text{if } 0\le r_Q(u,v)\cdot e^{i\omega_Q(u,v)}\le 1 \end{cases}$$
- $$\overline{t_Q(u,v)\cdot e^{i\theta_Q(u,v)}} = \begin{cases} \max(t_P(u), t_P(v))\cdot e^{i\max(\theta_P(u),\theta_P(v))} & \text{if } t_Q(u,v)\cdot e^{i\theta_Q(u,v)}=0 \\ \max(t_P(u), t_P(v))\cdot e^{i\max(\theta_P(u),\theta_P(v))} - t_Q(u,v)\cdot e^{i\theta_Q(u,v)} & \text{if } 0\le t_Q(u,v)\cdot e^{i\theta_Q(u,v)}\le 1 \end{cases}$$
- $$\overline{k_Q(u,v)\cdot e^{i\rho_Q(u,v)}} = \begin{cases} \max(k_P(u), k_P(v))\cdot e^{i\max(\rho_P(u),\rho_P(v))} & \text{if } k_Q(u,v)\cdot e^{i\rho_Q(u,v)}=0 \\ \max(k_P(u), k_P(v))\cdot e^{i\max(\rho_P(u),\rho_P(v))} - k_Q(u,v)\cdot e^{i\rho_Q(u,v)} & \text{if } 0\le k_Q(u,v)\cdot e^{i\rho_Q(u,v)}\le 1 \end{cases}$$

Example 3: The complement of CFNG G is represented in Figure 2 whose vertex set $\overline{P}$ is shown in Table 6 and edge set $\overline{Q}$ is shown in Table 7 and is denoted by CFNG $\overline{G}$.

Table 6. Vertices of CFNG $\overline{G}$

Node	CFNS
v_1	$\langle 0.5e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.2)}\rangle$
v_2	$\langle 0.8e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.9)}\rangle$
v_3	$\langle 0.3e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.8)}, 0.4e^{i2\pi(0.5)}\rangle$

Table 7. Edges of CFNG $\overline{G}$

Edge	CFNS
$e_{12}^{'}$	$\left\langle 0.2e^{i2\pi(0.1)}, 0.5e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.7)} \right\rangle$
$e_{23}^{'}$	$\left\langle 0.1e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.5)} \right\rangle$

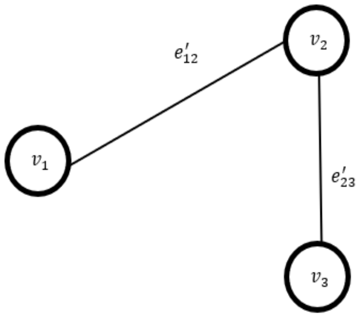

Figure 2. CFNG $\overline{G}$

Theorem 1: For a CFNG G , we have $\overline{\overline{G}} = G$.
Proof: Let G be a CFNG. Using definition 3.7 we have

$\overline{\overline{r_P(u)\cdot e^{l\omega_P(u)}}} = \overline{r_P(u)\cdot e^{l\omega_P(u)}} = r_P(u)\cdot e^{i\omega_P(u)}$,

$\overline{\overline{t_P(u)\cdot e^{l\theta_P(u)}}} = \overline{t_P(u)\cdot e^{l\theta_P(u)}} = t_P(u)\cdot e^{i\theta_P(u)}$

and $\overline{\overline{k_P(u)\cdot e^{l\rho_P(u)}}} = \overline{k_P(u)\cdot e^{l\rho_P(u)}} = k_P(u)\cdot e^{i\rho_P(u)}$ for all $u \in V$

$$\overline{\overline{r_Q(u,v)\cdot e^{l\omega_Q(u,v)}}} = \overline{r_Q(u,v)\cdot e^{l\omega_Q(u,v)}}$$

$$= \begin{cases} \min(r_P(u), r_P(v))\cdot e^{i\min(\omega_P(u),\omega_P(v))} & if\ r_Q(u,v)\cdot e^{i\omega_Q(u,v)} = 0 \\ \min(r_P(u), r_P(v))\cdot e^{i\min(\omega_P(u),\omega_P(v))} - r_Q(u,v)\cdot e^{i\omega_Q(u,v)} & if\ 0 \le r_Q(u,v)\cdot e^{i\omega_Q(u,v)} \le 1 \end{cases}$$

$$\overline{\overline{t_Q(u,v)\cdot e^{l\theta_Q(u,v)}}} = \overline{t_Q(u,v)\cdot e^{l\theta_Q(u,v)}}$$

$$= \begin{cases} \max(t_P(u), t_P(v))\cdot e^{i\max(\theta_P(u),\theta_P(v))} & if\ t_Q(u,v)\cdot e^{i\theta_Q(u,v)} = 0 \\ \max(t_P(u), t_P(v))\cdot e^{i\max(\theta_P(u),\theta_P(v))} - t_Q(u,v)\cdot e^{i\theta_Q(u,v)} & if\ 0 \le t_Q(u,v)\cdot e^{i\theta_Q(u,v)} \le 1 \end{cases}$$

$$\overline{\overline{k_Q(u,v)\cdot e^{l\rho_Q(u,v)}}} = \overline{k_Q(u,v)\cdot e^{l\rho_Q(u,v)}}$$

$$= \begin{cases} \max(k_P(u), k_P(v))\cdot e^{i\max(\rho_P(u),\rho_P(v))} & if\ k_Q(u,v)\cdot e^{i\rho_Q(u,v)} = 0 \\ \max(k_P(u), k_P(v))\cdot e^{i\max(\rho_P(u),\rho_P(v))} - t_Q(u,v)\cdot e^{i\rho_Q(u,v)} & if\ 0 \le k_Q(u,v)\cdot e^{i\rho_Q(u,v)} \le 1 \end{cases}$$

$(u,v) \in E$

Definition 3.9***:*** For $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$, the union between two CFNGS is given by

- $$\left(r_{P_1} \cup r_{P_2}\right)(u) \cdot e^{i\left(\omega_{P_1} \cup \omega_{P_2}\right)(u)} = \begin{cases} r_{P_1}(u) \cdot e^{i\left(\omega_{P_1}\right)(u)}, \ if \ u \in P_1 - P_2 \\ r_{P_2}(u) \cdot e^{i\left(\omega_{P_2}\right)(u)}, \ if \ u \in P_2 - P_1 \\ \max\left(r_{P_1}(u), r_{P_2}(u)\right) \cdot e^{i\max\left(\omega_{P_1}(u), \omega_{P_2}(u)\right)} \ if \ u \in P_1 \cap P_2 \end{cases}$$

- $$\left(t_{P_1} \cup t_{P_2}\right)(u) \cdot e^{i\left(\theta_{P_1} \cup \theta_{P_2}\right)(u)} = \begin{cases} t_{P_1}(u) \cdot e^{i\left(\theta_{P_1}\right)(u)}, \ if \ u \in P_1 - P_2 \\ t_{P_2}(u) \cdot e^{i\left(\theta_{P_2}\right)(u)}, \ if \ u \in P_2 - P_1 \\ \min\left(t_{P_1}(u), t_{P_2}(u)\right) \cdot e^{i\min\left(\theta_{P_1}(u), \theta_{P_2}(u)\right)} \ if \ u \in P_1 \cap P_2 \end{cases}$$

- $$\left(k_{P_1} \cup k_{P_2}\right)(u) \cdot e^{i\left(\rho_{P_1} \cup \rho_{P_2}\right)(u)} = \begin{cases} k_{P_1}(u) \cdot e^{i\left(\rho_{P_1}\right)(u)}, \ if \ u \in P_1 - P_2 \\ k_{P_2}(u) \cdot e^{i\left(\rho_{P_2}\right)(u)}, \ if \ u \in P_2 - P_1 \\ \min\left(k_{P_1}(u), k_{P_2}(u)\right) \cdot e^{i\min\left(\rho_{P_1}(u), \rho_{P_2}(u)\right)} \ if \ u \in P_1 \cap P_2 \end{cases}$$

- $$\left(r_{Q_1} \cup r_{Q_2}\right)(u,v) \cdot e^{i\left(\omega_{Q_1} \cup \omega_{Q_2}\right)(u,v)} = \begin{cases} \left(r_{Q_1}\right)(u,v) \cdot e^{i\left(\omega_{Q_1}\right)(u,v)} & if \ (u,v) \in Q_1 - Q_2 \\ \left(r_{Q_2}\right)(u,v) \cdot e^{i\left(\omega_{Q_2}\right)(u,v)} & if \ (u,v) \in Q_2 - Q_1 \\ \max\left(r_{Q_1}(u,v), r_{Q_2}(u,v)\right) \cdot e^{i\max\left(\omega_{Q_1}(u,v), \omega_{Q_2}(u,v)\right)} & if \ (u,v) \in Q_1 \cap Q_2 \end{cases}$$

- $$\left(t_{Q_1} \cup t_{Q_2}\right)(u,v) \cdot e^{i\left(\theta_{Q_1} \cup \theta_{Q_2}\right)(u,v)} = \begin{cases} \left(t_{Q_1}\right)(u,v) \cdot e^{i\left(\theta_{Q_1}\right)(u,v)} & if \ (u,v) \in Q_1 - Q_2 \\ \left(t_{Q_2}\right)(u,v) \cdot e^{i\left(\theta_{Q_2}\right)(u,v)} & if \ (u,v) \in Q_2 - Q_1 \\ \min\left(t_{Q_1}(u,v), t_{Q_2}(u,v)\right) \cdot e^{i\min\left(\theta_{Q_1}(u,v), \theta_{Q_2}(u,v)\right)} & if \ (u,v) \in Q_1 \cap Q_2 \end{cases}$$

- $$\left(k_{Q_1} \cup k_{Q_2}\right)(u,v) \cdot e^{i\left(\rho_{Q_1} \cup \rho_{Q_2}\right)(u,v)} = \begin{cases} \left(k_{Q_1}\right)(u,v) \cdot e^{i\left(\rho_{Q_1}\right)(u,v)} & if \ (u,v) \in Q_1 - Q_2 \\ \left(k_{Q_2}\right)(u,v) \cdot e^{i\left(\rho_{Q_2}\right)(u,v)} & if \ (u,v) \in Q_2 - Q_1 \\ \min\left(k_{Q_1}(u,v), k_{Q_2}(u,v)\right) \cdot e^{i\min\left(\rho_{Q_1}(u,v), \rho_{Q_2}(u,v)\right)} & if \ (u,v) \in Q_1 \cap Q_2 \end{cases}$$

Example 4: Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two CFNGs whose vertex sets and edge sets are provided in Table 8, Table 9, Table 10 and Table 11 and are represented in Figure 3 and Figure 4. respectively.

Table 8. Vertices of CFNG G_1

Node	CFNS
v_1	$\left\langle 0.5e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.6)}, 0.4^{i2\pi(0.6)} \right\rangle$
v_2	$\left\langle 0.8e^{i2\pi(0.3)}, 0.6e^{i2\pi(0.6)}, 0.4^{i2\pi(0.8)} \right\rangle$
v_4	$\left\langle 0.3e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.5)}, 0.6^{i2\pi(0.4)} \right\rangle$

Table 9. Edges of CFNG G_1

Edge	CFNS
e_{12}	$\left\langle 0.6e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.7)}, 0.7^{i2\pi(0.9)} \right\rangle$
e_{14}	$\left\langle 0.4e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.8)}, 0.6^{i2\pi(0.8)} \right\rangle$

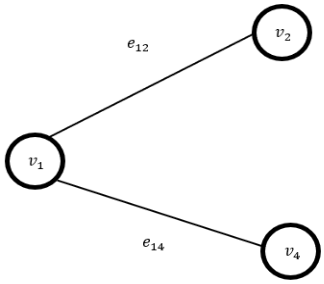

Figure 3. CFNG G_1

Table 10. Vertices of CFNG G_2

Node	CFNS
v_1	$\left\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.5^{i2\pi(0.6)} \right\rangle$
v_2	$\left\langle 0.8e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.8)}, 0.6^{i2\pi(0.4)} \right\rangle$
v_3	$\left\langle 0.6e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.7)}, 0.3^{i2\pi(0.5)} \right\rangle$
v_4	$\left\langle 0.3e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.9)}, 0.4^{i2\pi(0.3)} \right\rangle$

Table 11. Edges of CFNG G_2

Edge	CFNS
e_{12}	$\left\langle 0.4e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.7)}, 0.6^{i2\pi(0.4)} \right\rangle$
e_{13}	$\left\langle 0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.6)}, 0.5^{i2\pi(0.3)} \right\rangle$
e_{24}	$\left\langle 0.5e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.7)}, 0.6^{i2\pi(0.8)} \right\rangle$
e_{34}	$\left\langle 0.7e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.5)}, 0.4^{i2\pi(0.3)} \right\rangle$

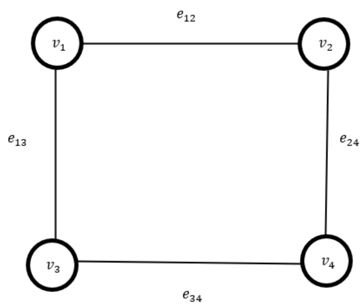

Figure 4. CFNG G_1

Then by definition 3.9, $v_3 \in P_2 \cap P_1$ and $v_1, v_2, v_4 \in P_1 \cap P_2$. The union of two CFNGs $G_1 \cup G_2$ is shown in Figure 5 whose vertex set and edge set given in Table 12 and Table 13 respectively.

Table 12. Vertices of $G_1 \cup G_2$

Node	CFNS
v_1	$\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.4^{i2\pi(0.6)} \rangle$
v_2	$\langle 0.8e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.6)}, 0.4^{i2\pi(0.4)} \rangle$
v_3	$\langle 0.6e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.7)}, 0.3^{i2\pi(0.5)} \rangle$
v_4	$\langle 0.3e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.5)}, 0.4^{i2\pi(0.3)} \rangle$

Table 13. Edges of $G_1 \cup G_2$

Edge	CFFS
e_{12}	$\langle 0.6e^{i2\pi(0.8)}, 0.3e^{i2\pi(0.7)}, 0.6^{i2\pi(0.4)} \rangle$
e_{13}	$\langle 0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.6)}, 0.5^{i2\pi(0.3)} \rangle$
e_{14}	$\langle 0.4e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.8)}, 0.6^{i2\pi(0.8)} \rangle$
e_{24}	$\langle 0.5e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.7)}, 0.6^{i2\pi(0.8)} \rangle$
e_{34}	$\langle 0.7e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.5)}, 0.4^{i2\pi(0.3)} \rangle$

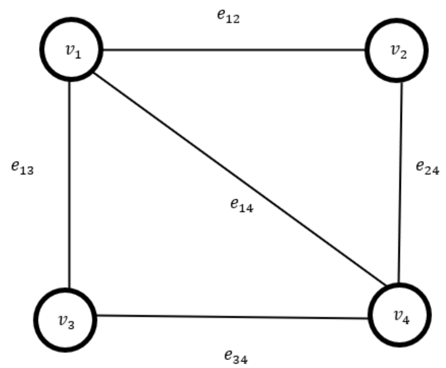

Figure 5. CFNG $G_1 \cup G_2$

Definition 3.10: Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two CFNSs, then the ring-sum $G_1 \oplus G_2 = (P_1 \oplus P_2, Q_1 \oplus Q_2)$ is defined as follows:

$$\begin{cases} (r_{P_1} \oplus r_{P_2})(u) \cdot e^{i(\omega_{P_1} \oplus \omega_{P_2})(u)} = (r_{P_1} \cup r_{P_2})(u) \cdot e^{i(\omega_{P_1} \cup \omega_{P_2})(u)} \\ (t_{P_1} \oplus t_{P_2})(u) \cdot e^{i(\theta_{P_1} \oplus \theta_{P_2})(u)} = (t_{P_1} \cup t_{P_2})(u) \cdot e^{i(\theta_{P_1} \cup \theta_{P_2})(u)} \\ (k_{P_1} \oplus k_{P_2})(u) \cdot e^{i(\rho_{P_1} \oplus \rho_{P_2})(u)} = (k_{P_1} \cup k_{P_2})(u) \cdot e^{i(\rho_{P_1} \cup \rho_{P_2})(u)} \ if \ u \in P_1 \cup P_2 \end{cases}$$

$$(r_{Q_1} \oplus r_{Q_2})(u,v) \cdot e^{i(\omega_{Q_1} \oplus \omega_{Q_2})(u,v)} = \begin{cases} r_{Q_1}(u,v) \cdot e^{i(\omega_{Q_1})(u,v)}, & f \ (u,v) \in Q_1 - Q_2 \\ r_{Q_2}(u,v) \cdot e^{i(\omega_{Q_2})(u,v)}, & f \ (u,v) \in Q_2 - Q_1 \\ 0, & f \ (u,v) \in Q_1 \cap Q \end{cases}$$

$$\left(t_{Q_1} \oplus t_{Q_2}\right)(u,v)\cdot e^{i\left(\theta_{Q_1}\oplus\theta_{Q_2}\right)(u,v)} = \begin{cases} t_{Q_1}(u,v)\cdot e^{i\left(\theta_{Q_1}\right)(u,v)}, & f\ (u,v)\in Q_1 - Q_2 \\ t_{Q_2}(u,v)\cdot e^{i\left(\theta_{Q_2}\right)(u,v)}, & f\ (u,v)\in Q_2 - Q_1 \\ 0, & f\ (u,v)\in Q_1 \cap Q \end{cases}$$

$$\left(k_{Q_1} \oplus k_{Q_2}\right)(u,v)\cdot e^{i\left(\rho_{Q_1}\oplus\rho_{Q_2}\right)(u,v)} = \begin{cases} k_{Q_1}(u,v)\cdot e^{i\left(\rho_{Q_1}\right)(u,v)}, & f\ (u,v)\in Q_1 - Q_2 \\ k_{Q_2}(u,v)\cdot e^{i\left(\rho_{Q_2}\right)(u,v)}, & f\ (u,v)\in Q_2 - Q_1 \\ 0, & f\ (u,v)\in Q_1 \cap Q \end{cases}$$

Remark: If $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ are the CFNGs, then $G_1 \oplus G_2$ is the CFNG,

Definition 3.11: Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two CFNSs, then the join $G_1 + G_2 \ = (P_1 + P_2, Q_1 + Q_2)$ is defined as:

$$\begin{cases} \left(r_{P_1} + r_{P_2}\right)(u)\cdot e^{i\left(\omega_{P_1}+\omega_{P_2}\right)(u)} = \left(r_{P_1} \cup r_{P_2}\right)(u)\cdot e^{i\left(\omega_{P_1}\cup\omega_{P_2}\right)(u)} \\ \left(t_{P_1} + t_{P_2}\right)(u)\cdot e^{i\left(\theta_{P_1}+\theta_{P_2}\right)(u)} = \left(t_{P_1} \cup t_{P_2}\right)(u)\cdot e^{i\left(\theta_{P_1}\cup\theta_{P_2}\right)(u)} \\ \left(k_{P_1} + k_{P_2}\right)(u)\cdot e^{i\left(\rho_{P_1}+\rho_{P_2}\right)(u)} = \left(k_{P_1} \cup k_{P_2}\right)(u)\cdot e^{i\left(\rho_{P_1}\cup\rho_{P_2}\right)(u)} \ if\ u\in P_1\cup P_2 \end{cases}$$

$$\begin{cases} \left(r_{Q_1} \oplus r_{Q_2}\right)(u)\cdot e^{i\left(\omega_{Q_1}\oplus\omega_{Q_2}\right)(u)} = \left(r_{Q_1} \cup r_{Q_2}\right)(u)\cdot e^{i\left(\omega_{Q_1}\cup\omega_{Q_2}\right)(u)} \\ \left(t_{P_1} \oplus t_{P_2}\right)(u)\cdot e^{i\left(\theta_{P_1}\oplus\theta_{P_2}\right)(u)} = \left(t_{Q_1} \cup t_{Q_2}\right)(u)\cdot e^{i\left(\theta_{Q_1}\cup\theta_{PQ}\right)(u)} \\ \left(k_{Q_1} \oplus k_{Q_2}\right)(u)\cdot e^{i\left(\rho_{Q_1}\oplus\rho_{Q_2}\right)(u)} = \left(k_{Q_1} \cup k_{Q_2}\right)(u)\cdot e^{i\left(\rho_{Q_1}\cup\rho_{Q_2}\right)(u)} \ if\ u\in Q_1\cup Q_2 \end{cases}$$

$$\begin{cases} \left(r_{Q_1} \oplus r_{Q_2}\right)(u)\cdot e^{i\left(\omega_{Q_1}\oplus\omega_{Q_2}\right)(u)} = \min\left(r_{Q_1} \cup r_{Q_2}\right)(u)\cdot e^{i\min\left(\omega_{Q_1}\cup\omega_{Q_2}\right)(u)} \\ \left(t_{Q_1} \oplus t_{Q_2}\right)(u)\cdot e^{i\left(\theta_{Q_1}\oplus\theta_{Q_2}\right)(u)} = \min\left(t_{Q_1} \cup t_{Q_2}\right)(u)\cdot e^{i\min\left(\theta_{Q_1}\cup\theta_{PQ}\right)(u)} \\ \left(k_{Q_1} \oplus k_{Q_2}\right)(u)\cdot e^{i\left(\rho_{Q_1}\oplus\rho_{Q_2}\right)(u)} = \max\left(k_{Q_1} \cup k_{Q_2}\right)(u)\cdot e^{i\min\left(\rho_{Q_1}\cup\rho_{Q_2}\right)(u)} \ if\ u\in Q' \end{cases}$$

where Q' is the set of all edges joining the vertices of P_1 and P_2, $P_1 \cap P_2 = \varnothing$.

Theorem 2: The join $G_1 + G_2$ of G_1 and G_2 is a CFNG of $G_1 + G_2$ iff G_1 and G_2 are CFNGs of G_1 and G_2, where $P_1 \cap P_2 = \varnothing$.

Definition 3.12: For CFNG G, the degree of a vertex $u \in V$ is denoted by $D(u)$ and is defined by

$$D(u) = \left(D_{re^{i\omega}}(u), D_{te^{i\theta}}(u), D_{ke^{i\rho}}(u)\right)$$

Where $D_{re^{i\omega}}(u) = \sum_{(u,v)\neq u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\omega_{Q(u,v)}}$; $D_{te^{i\theta}}(u) = \sum_{(u,v)\neq u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\theta_{Q(u,v)}}$

$D_{ke^{i\rho}}(u) = \sum_{(u,v)\neq u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\rho_{Q(u,v)}}$

Definition 3.13*:* For CFNG G, the total degree of a vertex $u \in V$ is denoted by $TD(u)$ and is defined by

$$TD(u) = \left(TD_{re^{i\omega}}(u), TD_{te^{i\theta}}(u), TD_{ke^{i\rho}}(u)\right)\left(TD_r e^{i\omega(u)}(u), TD_k(u) e^{i\rho(u)}\right),$$

Where $TD_{re^{i\omega}}(u) = \sum_{(u,v)\neq u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\omega_{Q(u,v)}} + r_P(u)\cdot e^{i\omega_P(u)}$

$TD_{te^{i\theta}}(u) = \sum_{(u,v)\neq u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\theta_{Q(u,v)}} + t_P(u)\cdot e^{i\theta_P(u)}$

$TD_{ke^{i\rho}}(u) = \sum_{(u,v)\neq u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\rho_{Q(u,v)}} + k_P(u)\cdot e^{i\rho_P(u)}$

Definition 3.14: Let G_1 and G_2 be two CFNGs. For any vertex $u \in P_1 \cup P_2$, there are three cases to consider.

Case 1: Either $u \in P_1 - P_2$ or $u \in P_2 - P_1$. Then, no edge incident at u lies in $Q_1 \cup Q_2$. Thus, for $u \in P_1 - P_2$

$$(D_{re^{i\omega}})_{G_1 \cup G_2}(u) = \sum_{(u,v)\in Q_1} r_{Q_1}(u,v)\cdot e^{\sum_{(u,v)\in Q_1}\omega_{Q_1}(u,v)} = (D_{re^{i\omega}})_{G_1}(u)$$

$$(D_{te^{i\theta}})_{G_1 \cup G_2}(u) = \sum_{(u,v)\in Q_1} t_{Q_1}(u,v)\cdot e^{\sum_{(u,v)\in Q_1}\theta_{Q_1}(u,v)} = (D_{te^{i\theta}})_{G_1}(u)$$

$$(D_{ke^{i\rho}})_{G_1 \cup G_2}(u) = \sum_{(u,v)\in Q_1} k_{Q_1}(u,v)\cdot e^{\sum_{(u,v)\in Q_1}\rho_{Q_1}(u,v)} = (D_{te^{i\rho}})_{G_1}(u)$$

$$(TD_{ke^{i\rho}})_{G_1 \cup G_2}(u) = (TD_{re^{i\omega}})_{G_1}(u); (TD_{ke^{i\rho}})_{G_1 \cup G_2}(u) = (TD_{ke^{i\rho}})_{G_1}(u)$$

Case 2: Either $u \in P_1 \cap P_2$ but no edge incident at $u \in Q_1 \cap Q_2$. Then, any edge incident at u is either in $Q_1 - Q_2$ or $Q_2 - Q_1$.

$$(D_{re^{i\omega}})_{G_1 \cup G_2}(u) = (D_{re^{i\omega}})_{G_1}(u) + (D_{re^{i\omega}})_{G_2}(u)$$

$$(D_{te^{i\theta}})_{G_1 \cup G_2}(u) = (D_{te^{i\theta}})_{G_1}(u) + (D_{te^{i\theta}})_{G_2}(u)$$

$$(D_{ke^{i\rho}})_{G_1 \cup G_2}(u) = (D_{ke^{i\rho}})_{G_1}(u) + (D_{ke^{i\rho}})_{G_2}(u)$$

$$(TD_{re^{i\omega}})_{G_1 \cup G_2}(u) = (TD_{re^{i\omega}})_{G_1}(u) + (TD_{re^{i\omega}})_{G_2}(u) - \min(r_{P_1}(u), r_{P_2}(u))\cdot e^{i\min(\omega_{P_1}(u),\omega_{P_2}(u))}$$

$$(TD_{te^{i\theta}})_{G_1 \cup G_2}(u) = (TD_{te^{i\theta}})_{G_1}(u) + (TD_{te^{i\theta}})_{G_2}(u) - \min(t_{P_1}(u), t_{P_2}(u))\cdot e^{i\min(\theta_{P_1}(u),\theta_{P_2}(u))}$$

$$(TD_{ke^{i\rho}})_{G_1 \cup G_2}(u) = (TD_{ke^{i\rho}})_{G_1}(u) + (TD_{ke^{i\rho}})_{G_2}(u) - \min(k_{P_1}(u), k_{P_2}(u))\cdot e^{i\min(\rho_{P_1}(u),\rho_{P_2}(u))}$$

Case 3: $u \in P_1 \cap P_2$ and some edges incident at u are in $Q_2 - Q_1$.

$$(D_{re^{i\omega}})_{G_1 \cup G_2}(u)$$
$$= (D_{re^{i\omega}})_{G_1}(u) + (D_{re^{i\omega}})_{G_2}(u) - \sum_{(u,v)\in Q_1\cap Q_2} \min(r_{Q_1}(u,v), r_{Q_2}(u,v))\cdot e^{i\sum_{(u,v)\in Q_1\cap Q_2}\min(\omega_{Q_1}(u,v),\omega_{Q_2}(u,v))}$$

$$(D_{te^{i\theta}})_{G_1 \cup G_2}(u)$$
$$= (D_{te^{i\theta}})_{G_1}(u) + (D_{te^{i\theta}})_{G_2}(u) - \sum_{(u,v)\in Q_1\cap Q_2} \min(t_{Q_1}(u,v), t_{Q_2}(u,v))\cdot e^{i\sum_{(u,v)\in Q_1\cap Q_2}\min(\theta_{Q_1}(u,v),\theta_{Q_2}(u,v))}$$

$$(D_{ke^{i\rho}})_{G_1 \cup G_2}(u)$$
$$= (D_{ke^{i\rho}})_{G_1}(u) + (D_{ke^{i\rho}})_{G_2}(u) - \sum_{(u,v)\in Q_1\cap Q_2} \max(k_{Q_1}(u,v), k_{Q_2}(u,v))\cdot e^{i\sum_{(u,v)\in Q_1\cap Q_2}\max(\rho_{Q_1}(u,v),\rho_{Q_2}(u,v))}$$

$$(TD_{re^{i\omega}})_{G_1 \cup G_2}(u)$$
$$= (TD_{re^{i\omega}})_{G_1}(u) + (TD_{re^{i\omega}})_{G_2}(u) - \sum_{(u,v)\in Q_1\cap Q_2} \min(r_{Q_1}(u,v), r_{Q_2}(u,v))\cdot e^{i\sum_{(u,v)\in Q_1\cap Q_2}\min(\omega_{Q_1}(u,v),\omega_{Q_2}(u,v))}$$

$$(TD_{te^{i\theta}})_{G_1 \cup G_2}(u)$$
$$= (TD_{te^{i\theta}})_{G_1}(u) + (TD_{te^{i\theta}})_{G_2}(u) - \sum_{(u,v)\in Q_1\cap Q_2} \min(t_{Q_1}(u,v), t_{Q_2}(u,v))\cdot e^{i\sum_{(u,v)\in Q_1\cap Q_2}\min(\theta_{Q_1}(u,v),\theta_{Q_2}(u,v))}$$

$$(TD_{ke^{i\rho}})_{G_1 \cup G_2}(u)$$
$$= (TD_{ke^{i\rho}})_{G_1}(u) + (TD_{ke^{i\rho}})_{G_2}(u) - \sum_{(u,v)\in Q_1\cap Q_2} \max(k_{Q_1}(u,v), k_{Q_2}(u,v))\cdot e^{i\sum_{(u,v)\in Q_1\cap Q_2}\max(\rho_{Q_1}(u,v),\rho_{Q_2}(u,v))}$$

Definition 3.15: For any vertex $u \in P_1 \oplus P_2$ of two CFNGs G_1 and G_2, then two cases are to be considered. If

Case 1: $u \in P_1 - P_2$ or $u \in P_2 - P_1$

Case 2: $u \in P_1 \cap P_2$, then any edge incident at u is either in $Q_1 - Q_2$ or $Q_2 - Q_1$. For these cases we have

$$\left(D_{re^{i\omega}}\right)_{G_1 \oplus G_2}(u) = \left(D_{re^{i\omega}}\right)_{G_1 \cup G_2}, \left(D_{ke^{i\omega}}\right)_{G_1 \oplus G_2}(u) = \left(D_{ke^{i\omega}}\right)_{G_1 \cup G_2}(u)$$

$$\left(TD_{re^{i\omega}}\right)_{G_1 \oplus G_2}(u) = \left(TD_{re^{i\omega}}\right)_{G_1 \cup G_2}, \left(TD_{ke^{i\omega}}\right)_{G_1 \oplus G_2}(u) = \left(TD_{ke^{i\omega}}\right)_{G_1 \cup G_2}(u)$$

Definition 3.16: $G_1 \times G_2$ is the cartesian product of two CFNGs and is defined as a pair $G_1 \times G_2 = \left(P_1 \times P_2, Q_1 \times Q_2\right)$, such that

- $r_{P_1 \times P_2}(u,v) \cdot e^{i\omega_{P_1 \times P_2}(u,v)} = \min\left\{r_{P_1}(u), r_{P_2}(v)\right\} \cdot e^{i\min\left\{\omega_{P_1}(u), \omega_{P_2}(v)\right\}}$

  $t_{P_1 \times P_2}(u,v) \cdot e^{i\theta_{P_1 \times P_2}(u,v)} = \max\left\{t_{P_1}(u), t_{P_2}(v)\right\} \cdot e^{i\max\left\{\theta_{P_1}(u), \theta_{P_2}(v)\right\}}$

  $k_{P_1 \times P_2}(u,v) \cdot e^{i\rho_{P_1 \times P_2}(u,v)} = \max\left\{k_{P_1}(u), k_{P_2}(v)\right\} \cdot e^{i\max\left\{\rho_{P_1}(u), \rho_{P_2}(v)\right\}}$ for all $u, v \in V$.
- $r_{Q_1 \times Q_2}\left\langle (u,u_2),(u,v_2)\right\rangle \cdot e^{i\omega_{Q_1 \times Q_2}\left\langle (u,u_2),(u,v_2)\right\rangle} = \min\left\{r_{P_1}(u), r_{Q_2}(u_2,v_2)\right\} \cdot e^{i\min\left\{\omega_{P_1}(u), \omega_{Q_2}(u_2,v_2)\right\}}$

  $t_{Q_1 \times Q_2}\left\langle (u,u_2),(u,v_2)\right\rangle \cdot e^{i\theta_{Q_1 \times Q_2}\left\langle (u,u_2),(u,v_2)\right\rangle} = \max\left\{t_{P_1}(u), t_{Q_2}(u_2,v_2)\right\} \cdot e^{i\max\left\{\theta_{P_1}(u), \theta_{Q_2}(u_2,v_2)\right\}}$

  $k_{Q_1 \times Q_2}\left\langle (u,u_2),(u,v_2)\right\rangle \cdot e^{i\rho_{Q_1 \times Q_2}\left\langle (u,u_2),(u,v_2)\right\rangle} = \max\left\{k_{P_1}(u), k_{Q_2}(u_2,v_2)\right\} \cdot e^{i\max\left\{\rho_{P_1}(u), \rho_{Q_2}(u_2,v_2)\right\}}$

  for all $u \in V_1, (u_2,v_2) \in E_2$
- $r_{Q_1 \times Q_2}\left\langle (u_1,w),(v_1,w)\right\rangle \cdot e^{i\omega_{Q_1 \times Q_2}\left\langle (u_1,w),(v_1,w)\right\rangle} = \min\left\{r_{P_1}(u_1,v_1), r_{Q_2}(w)\right\} \cdot e^{i\min\left\{\omega_{P_1}(u_1,v_1), \omega_{Q_2}(w)\right\}}$

  $t_{Q_1 \times Q_2}\left\langle (u_1,w),(v_1,w)\right\rangle \cdot e^{i\theta_{Q_1 \times Q_2}\left\langle (u_1,w),(v_1,w)\right\rangle} = \max\left\{t_{P_1}(u_1,v_1), t_{Q_2}(w)\right\} \cdot e^{i\max\left\{\theta_{P_1}(u_1,v_1), \theta_{Q_2}(w)\right\}}$

  $k_{Q_1 \times Q_2}\left\langle (u_1,w),(v_1,w)\right\rangle \cdot e^{i\rho_{Q_1 \times Q_2}\left\langle (u_1,w),(v_1,w)\right\rangle} = \max\left\{k_{P_1}(u_1,v_1), k_{Q_2}(w)\right\} \cdot e^{i\max\left\{\rho_{P_1}(u_1,v_1), \rho_{Q_2}(w)\right\}}$

  for all $w \in V_2, (u_1,v_1) \in E_1$

3.3. Regular and totally regular CFNGs

Regular and totally regular CFNGs are introduced in this section along with examples.

Definition 3.17: A CFNG $G = (P,Q)$ is with each vertex having same degree is said to be a regular CFNG. If the degree of each vertex $D_G(u_i) = \left(D_{re^{i\omega}}(u), D_{te^{i\theta}}(u), D_{ke^{i\rho}}(u)\right) = (u_1,u_2,u_3)$ for all $u_i \in V$, where

$$D_{re^{i\omega}}(u) = \sum_{(u,v)\neq u\in V} r_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in V}\omega_Q(u,v)} = u_1$$

$$D_{te^{i\theta}}(u) = \sum_{(u,v)\neq u\in V} t_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in V}\theta_Q(u,v)} = u_2$$

$$D_{ke^{i\rho}}(u) = \sum_{(u,v)\neq u\in V} k_Q(u,v) \cdot e^{\sum_{(u,v)\neq u\in V}\rho_Q(u,v)} = u_3$$

G is called regular of degree (u_1,u_2,u_3) or (u_1,u_2,u_3)-regular.

Example 5: Consider a CFNG $G = (P,Q)$ on $P = \{v_1,v_2,v_3,v_4\}$ as in Figure 6 whose vertex and edge set are given in Table 14 and Table 15 respectively.

Table 14. Vertices of regular CFNG G

Node	CFNS
v_1	$\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.5^{i2\pi(0.6)} \rangle$
v_2	$\langle 0.8e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.8)}, 0.6^{i2\pi(0.4)} \rangle$
v_3	$\langle 0.6e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.7)}, 0.3^{i2\pi(0.5)} \rangle$
v_4	$\langle 0.3e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.9)}, 0.4^{i2\pi(0.3)} \rangle$

Table 15. Edges of regular CFNG G

Edge	CFNS
e_{12}	$\langle 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.4)}, 0.5^{i2\pi(0.4)} \rangle$
e_{13}	$\langle 0.2e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.3)}, 0.4^{i2\pi(0.8)} \rangle$
e_{24}	$\langle 0.2e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.3)}, 0.4^{i2\pi(0.8)} \rangle$
e_{34}	$\langle 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.4)}, 0.5^{i2\pi(0.4)} \rangle$

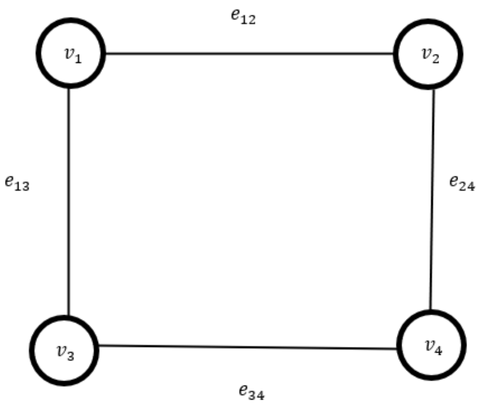

Figure 6. Regular CFNG G

Then the degrees of vertices v_1, v_2, v_3, v_4 are determined as:

$$D_G(v_1) = \left(r_Q(v_1, \mathrm{v}_2) + r_Q(v_1, \mathrm{v}_3)\right) \cdot e^{i\left(\omega_Q(v_1,\mathrm{v}_2)+\omega_Q(v_1,\mathrm{v}_2)\right)},$$
$$\left(t_Q(v_1, \mathrm{v}_2) + t_Q(v_1, \mathrm{v}_3)\right) \cdot e^{i\left(\theta_Q(v_1,\mathrm{v}_2)+\theta_Q(v_1,\mathrm{v}_2)\right)}, \left(k_Q(v_1, \mathrm{v}_2) + k_Q(v_1, \mathrm{v}_3)\right) \cdot e^{i\left(\rho_Q(v_1,\mathrm{v}_2)+\rho_Q(v_1,\mathrm{v}_2)\right)},$$
$$= \langle 0.5^{i2\pi(1.1)}, 0.6^{i2\pi(0.7)}, 0.9^{i2\pi(1.2)} \rangle$$

Similarly, $D_G(v_2) = D_G(v_3) = D_G(v_4) = \langle 0.5^{i2\pi(1.1)}, 0.6^{i2\pi(0.7)}, 0.9^{i2\pi(1.2)} \rangle$. Hence, G is $\langle 0.5^{i2\pi(1.1)}, 0.6^{i2\pi(0.7)}, 0.9^{i2\pi(1.2)} \rangle$ - regular graph.

Definition 3.18: A CFNG $G = (P, Q)$ on G is called a totally regular CFNG if the total degree of its each vertex is same. If each vertex has total degree (m_1, m_2, m_3), that is, $TD_G(u_i) = \left(TD_{re^{i\omega}}(u), TD_{te^{i\theta}}(u), TD_{ke^{i\rho}}(u)\right) = (m_1, m_2, m_3)$ for all $u_i \in V$, where

$$TD_{re^{i\omega}}(u) = \sum_{(u,v)\neq u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\omega_Q(u,v)} + r_P(u)\cdot e^{i\omega_P(u)} = m_1$$
$$TD_{te^{i\theta}}(u) = \sum_{(u,v)\neq u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\theta_Q(u,v)} + t_P(u)\cdot e^{i\theta_P(u)} = m_2$$
$$TD_{ke^{i\rho}}(u) = \sum_{(u,v)\neq u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\rho_Q(u,v)} + k_P(u)\cdot e^{i\rho_P(u)} = m_3$$

G is called totally regular of degree (m_1, m_2, m_3) or (m_1, m_2, m_3) -totally regular.

Example 6: Consider a CFNG $G = (P, Q)$ on $P = \{v_1, v_2, v_3, v_4\}$ as in Figure 7 whose vertex and edge set are given in Table 16 and Table 17 respectively.

Table 16. Vertices of totally regular CFNG G

Node	CFNS
v_1	$\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.5^{i2\pi(0.6)} \rangle$
v_2	$\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.5^{i2\pi(0.6)} \rangle$
v_3	$\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.5^{i2\pi(0.6)} \rangle$
v_4	$\langle 0.7e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.3)}, 0.5^{i2\pi(0.6)} \rangle$

Table 17. Edges of totally regular CFNG G

Edge	CFNS
e_{12}	$\langle 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.4)}, 0.5^{i2\pi(0.4)} \rangle$
e_{13}	$\langle 0.2e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.3)}, 0.4^{i2\pi(0.8)} \rangle$
e_{24}	$\langle 0.2e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.3)}, 0.4^{i2\pi(0.8)} \rangle$
e_{34}	$\langle 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.4)}, 0.5^{i2\pi(0.4)} \rangle$

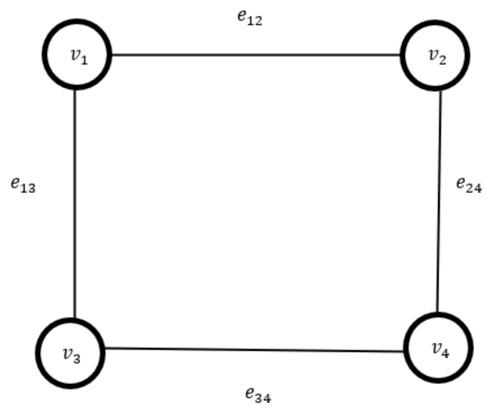

Figure 7. Totally regular CFNG G

$TD_G(v_1) = \big(r_Q(v_1, v_2) + r_Q(v_1, v_3) + r_P(v_1)\big)\cdot e^{i(\omega_Q(v_1,v_2)+\omega_Q(v_1,v_2)+\omega_P(v_1))}$,
$\big(t_Q(v_1, v_2) + t_Q(v_1, v_3) + t_P(v_1)\big)\cdot e^{i(\theta_Q(v_1,v_2)+\theta_Q(v_1,v_2)+\theta_p(v_1,))}$, ,
$\big(k_Q(v_1, v_2) + k_Q(v_1, v_3) + k_P(v_1)\big)\cdot e^{i(\rho_Q(v_1,v_2)+\rho_Q(v_1,v_2)+\rho_P(v_1))}$
$= \langle 1.2^{i2\pi(1.6)}, 1.2^{i2\pi(1.0)}, 1.4^{i2\pi(1.8)} \rangle$

Similarly, $TD_G(v_2)=TD_G(v_3)=TD_G(v_4)=\langle 1.2^{i2\pi(1.6)},1.2^{i2\pi(1.0)},1.4^{i2\pi(1.8)}\rangle$. Hence G is $\langle 1.2^{i2\pi(1.6)},1.2^{i2\pi(1.0)},1.4^{i2\pi(1.8)}\rangle$ - totally regular CFNG.

Theorem 3: Let $G=(P,Q)$ be a (u_1,u_2,u_3)-regular CFNG. Then $S(G)=\left(\frac{nu_1}{2},\frac{nn_2}{2},\frac{nn_3}{2}\right)$ where $|V|=n$.

Proof: Assume that G is a CFNG with size

$$S(G)=\left(\sum_{(u,v)\in E} r_Q(u,v)\cdot e^{\sum_{(u,v)\in E}\omega_Q(u,v)},\sum_{(u,v)\in E} t_Q(u,v)\cdot e^{\sum_{(u,v)\in E}\theta_Q(u,v)},\sum_{(u,v)\in E} k_Q(u,v)\cdot e^{\sum_{(u,v)\in E}\rho_Q(u,v)}\right)$$

Since $G(u_1,u_2,u_3)$-regular CFNG, that is

$D_G(u)_i=\left(D_{re^{i\omega}}(u),D_{te^{i\theta}}(u),D_{ke^{i\rho}}(u)\right)=(u_1,u_2,u_3)$ for all $u_i\in V$, where

$$D_{re^{i\omega}}(u)=\sum_{(u,v)\neq u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\omega_Q(u,v)}=u_1$$
$$D_{te^{i\theta}}(u)=\sum_{(u,v)\neq u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\theta_Q(u,v)}=u_2$$
$$D_{ke^{i\rho}}(u)=\sum_{(u,v)\neq u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\rho_Q(u,v)}=u_3$$

Therefore,

$$D_G(u_i)=\left(\sum_{(u,v)\in E} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\omega_Q(u,v)},\sum_{(u,v)\in E} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\theta_Q(u,v)},\sum_{(u,v)\in E} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\rho_Q(u,v)}\right)$$

$$\sum_{u_i\in V} D_G(u_i)=\left(\sum_{u_i\in V}\sum_{(u,v)\in E} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\omega_Q(u,v)},\sum_{u_i\in V}\sum_{(u,v)\in E} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\theta_Q(u,v)},\sum_{u_i\in V}\sum_{(u,v)\in E} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\rho_Q(u,v)}\right)$$

Since each edge is considered twice, so

$$\left(\sum_{u_i\in V} D_{re^{i\omega}}(u_i),\sum_{u_i\in V} D_{te^{i\theta}}(u_i),\sum_{u_i\in V} D_{ke^{i\rho}}(u_i)\right)=2\left(\sum_{(u,v)\in E} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\omega_Q(u,v)},\sum_{(u,v)\in E} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\theta_Q(u,v)},\sum_{(u,v)\in E} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\rho_Q(u,v)}\right)$$

$$(nu_1,nu_2,nu_3)=2S(G)$$
$$S(G)=\left(\frac{nu_1}{2},\frac{nn_2}{2},\frac{nn_3}{2}\right)$$

Theorem 4: Let $G=(P,Q)$ be a (m_1,m_2,m_3) totally regular graph CFNG. Then $2S(G)+O(G)=(nm_1,nm_2,nm_3)$ where $|V|-n$.

$$TD_{re^{i\omega}}(u)=\sum_{(u,v)\neq u\in V} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\omega_Q(u,v)}+r_P(u)\cdot e^{i\omega_P(u)}=m_1$$
$$TD_{te^{i\theta}}(u)=\sum_{(u,v)\neq u\in V} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\theta_Q(u,v)}+t_P(u)\cdot e^{i\theta_P(u)}=m_2$$
$$TD_{ke^{i\rho}}(u)=\sum_{(u,v)\neq u\in V} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in V}\rho_Q(u,v)}+k_P(u)\cdot e^{i\rho_P(u)}=m_3$$

Therefore,

$$TD_G(u_i)=\begin{pmatrix}\sum_{(u,v)\neq u\in E} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\omega_Q(u,v)}+r_P(u)\cdot e^{i\omega_P(u)},\\ \sum_{(u,v)\neq u\in E} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\theta_Q(u,v)}+t_P(u)\cdot e^{i\theta_P(u)},\\ \sum_{(u,v)\neq u\in E} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\rho_Q(u,v)}+k_P(u)\cdot e^{i\rho_P(u)}\end{pmatrix}$$

$$\sum_{u_i\in V} TD_G(u_i)=\begin{pmatrix}\sum_{u_i\in V}\sum_{(u,v)\neq u\in E} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\omega_Q(u,v)}+r_P(u)\cdot e^{i\omega_P(u)},\\ \sum_{u_i\in V}\sum_{(u,v)\neq u\in E} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\theta_Q(u,v)}+t_P(u)\cdot e^{i\theta_P(u)},\\ \sum_{u_i\in V}\sum_{(u,v)\neq u\in E} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\rho_Q(u,v)}+k_P(u)\cdot e^{i\rho_P(u)}\end{pmatrix}$$

Since each edge is considered twice, so

$$\left(D_{re^{i\omega}}(u_i),D_{te^{i\theta}}(u_i),D_{ke^{i\rho}}(u_i)\right)=2\begin{pmatrix}\sum_{(u,v)\neq u\in E} r_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\omega_Q(u,v)},\ \sum_{(u,v)\neq u\in E} t_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\theta_Q(u,v)},\\ \sum_{(u,v)\neq u\in E} k_Q(u,v)\cdot e^{\sum_{(u,v)\neq u\in E}\rho_Q(u,v)}\end{pmatrix}$$
$$+\left(r_P(u)\cdot e^{i\omega_P(u)},t_P(u)\cdot e^{i\theta_P(u)},k_P(u)\cdot e^{i\rho_P(u)}\right)$$

Corollary 1: Let $G=(P,Q)$ be a (n_1,n_2,n_3)- regular and (m_1,m_2,m_3)- totally regular CFNG. Then $O(G)=n\{(m_1,m_2,m_3)-(n_1,n_2,n_3)\}$.

Proof: Assume that G is a (n_1,n_2,n_3)- regular CFNG. Then the size of G is

$$S(G)=\left(\frac{nu_1}{2},\frac{nn_2}{2},\frac{nn_3}{2}\right)$$

As G is a (m_1,m_2,m_3)- totally regular CFNG. Using theorem 4, we have

$$2S(G)+O(G)=(ng_1,ng_2,ng_3)$$
$$O(G)=(nm_1,nm_2,nm_3)-2S(G)$$
$$=(nm_1,nm_2,nm_3)-2\left(\frac{nu_1}{2},\frac{nn_2}{2},\frac{nn_3}{2}\right)$$
$$=n\{(m_1,m_2,m_3)-(n_1,n_2,n_3)\}.$$

4. Decision making approach under CFN environment

In this section, based on the proposed CFNGs a decision-making approach is introduced. Further a real-life application on education system is discussed.

Definition 4.1: Let $F_j=\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\rangle$ $(j=1,2,\cdots,m)$ be a collection of CFNNs and $w=(w_1,w_2,\cdots,w_n)^T$ be the weighted vector of F_j $(j=1,2,\cdots,m)$ with $w_j>0,\sum_{j=1}^{n}w_j=1$. Then the complex fermatean neutrosophic weighted averaging (CFNWA) operator is a mapping $CFNWA:\Omega^n\to\Omega,$ such that

$$CFNWA\left(F_1,F_2,\cdots,F_m\right)=\overset{m}{\underset{j=1}{\oplus}}\left(w_jF_j\right)$$

Specifically, if we have $w_j=\frac{1}{m};\forall j,$ then CFNWA reduces to complex Fermatean neutrosophic averaging (CFNA) operator

$$CFNA\left(F_1,F_2,\cdots,F_m\right)=\frac{1}{m}\overset{m}{\underset{j=1}{\oplus}}\left(w_jF_j\right)$$

We get CFNWA operator using induction on n and is defined as

$$CFNWA\left(F_1,F_2,\cdots,F_m\right)=\left(\begin{array}{c}\sqrt[3]{1-\prod_{j=1}^{m}\left(1-r_j^3\right)^{wj}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\omega_i}{2\pi}}\right)^{wj}},\\ \prod_{j=1}^{m}t_j^{w_j}\cdot e^{i2\pi\left(\left(\frac{\theta_j}{2\pi}\right)^{w_j}\right)},\prod_{j=1}^{m}k_j^{w_j}\cdot e^{i2\pi\left(\left(\frac{\pi_j}{2\pi}\right)^{w_j}\right)}\end{array}\right)$$

Definition 4.2: Let $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ be a collection of CFNNs and $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the weighted vector of $F_j\left(j=1,2,\cdots,m\right)$ with $w_j>0,\sum_{j=1}^{n}w_j=1.$ Then the complex fermatean neutrosophic weighted geometric (CFNWG) operator is defined by $CFNWG:\Omega^n\rightarrow\Omega,$ such that

$$CFNWG\left(F_1,F_2,\cdots,F_m\right)=\overset{m}{\underset{j=1}{\otimes}}\left(w_jF_j\right)$$

Theorem 5: The aggregated value of CFNNs $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ using CFNWG operator is again a CFNN and is given by

$$CFNWG\left(F_1,F_2,\cdots,F_m\right)=\left(\begin{array}{c}\prod_{j=1}^{m}r_j^{w_j}\cdot e^{i2\pi\left(\left(\frac{\omega_j}{2\pi}\right)^{w_j}\right)},\sqrt[3]{1-\prod_{i=1}^{m}\left(1-t_j^3\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\theta_j}{2\pi}}\right)^{w_j}},\\ \sqrt[3]{1-\prod_{j=1}^{m}\left(1-k_j^3\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\rho_j}{2\pi}}\right)^{w_j}}\end{array}\right)$$

Definition 4.3: Let $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ be a collection of CFNNs and $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the weighted vector of $F_j\left(j=1,2,\cdots,m\right)$ with $w_j>0,\sum_{j=1}^{n}w_j=1.$ Then the complex fermatean neutrosophic ordered weighted averaging (CFNOWA) operator is a mapping $CFNOWA:\Omega^n\rightarrow\Omega,$ such that

$$CFNOWA\left(F_1,F_2,\cdots,F_m\right)=\overset{m}{\underset{j=1}{\oplus}}\left(w_jF_{\sigma(j)}\right)$$

Where $\left(\sigma(1),\sigma(2),\cdots,\sigma(m)\right)$ denotes the permutation of $\left(1,2,\cdots,m\right)$ and $F_{\sigma(j-1)}\geq F_{\sigma(j)},\forall j=1,2,\cdots,m$. Using induction on n, we get the following theorem.

Theorem 6: The aggregated value of CFNNs $F_j=\left\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\right\rangle\ \left(j=1,2,\cdots,m\right)$ using CFNOWA is again a CFNN and is given by

$$CFNOWA(F_1,F_2,\cdots,F_m)=\left(\begin{array}{c}\sqrt[3]{1-\prod_{j=1}^{m}\left(1-r_{\sigma(j)}^3\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\omega_{\sigma(j)}}{2\pi}}\right)^{w_j}},\\ \prod_{j=1}^{m}t_{\sigma(j)}^{w_j}\cdot e^{i2\pi\left(\left(\frac{\theta_{\sigma(j)}}{2\pi}\right)^{w_j}\right)},\prod_{j=1}^{m}k_{\sigma(j)}^{w_j}\cdot e^{i2\pi\left(\left(\frac{\pi_{\sigma(j)}}{2\pi}\right)^{w_j}\right)}\end{array}\right)$$

In which $(\sigma(1),\sigma(2),\cdots,\sigma(m))$ denotes the permutation of $(1,2,\ldots,m)$ and $F_{\sigma(j-1)}\geq F_{\sigma(j)},\forall j=1,2,\cdots,m$ and $\sum_{j=1}^{n}w_j=1$.

Definition 4.4: Let $F_j=\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\rangle$ $(j=1,2,\cdots,m)$ be a collection of CFNNs and $w=(w_1,w_2,\cdots,w_n)^T$ be the weighted vector of F_j $(j=1,2,\cdots,m)$ with $w_j>0,\sum_{j=1}^{n}w_j=1$. Then the complex fermatean neutrosophic ordered weighted geometric (CFNOWG) operator is a mapping $CFNOWG:\Omega^n\to\Omega$, such that

$$CFNOWG(F_1,F_2,\cdots,F_m)=\bigotimes_{j=1}^{m}\left(w_jF_{\sigma(j)}\right)$$

Where $(\sigma(1),\sigma(2),\cdots,\sigma(m))$ denotes the permutation of $(1,2,\cdots,m)$ and $F_{\sigma(j-1)}\geq F_{\sigma(j)},\forall j=1,2,\cdots,m$. Using induction on n, we get the following theorem.

Theorem 7: The aggregated value of CFNNs
$F_j=\langle r_j\cdot e^{i2\pi\omega},t_j\cdot e^{i2\pi\theta},k_j\cdot e^{i2\pi\rho}\rangle$ $(j=1,2,\cdots,m)$ using CFNOWGO is again a CFNN and is given by

$$CFNOWG(F_1,F_2,\cdots,F_m)=\left(\begin{array}{c}\prod_{j=1}^{m}r_{\sigma(j)}^{w_j}\cdot e^{i2\pi\left(\left(\frac{\omega_{\sigma(j)}}{2\pi}\right)^{w_j}\right)},\sqrt[3]{1-\prod_{j=1}^{m}\left(1-t_{\sigma(j)}^3\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\theta_{\sigma(j)}}{2\pi}}\right)^{w_j}},\\ \sqrt[3]{1-\prod_{j=1}^{m}\left(1-k_{\sigma(j)}^3\right)^{w_j}}\cdot e^{i2\pi\left(\sqrt[3]{1-\prod_{j=1}^{m}1-\frac{\rho_{\sigma(j)}}{2\pi}}\right)^{w_j}}\end{array}\right)$$

Now, we propose the idea of score function and accuracy function in order to compare two CFNN.

Definition 4.5: Let $A=\langle r\cdot e^{i2\pi\omega},t\cdot e^{i2\pi\theta},k\cdot e^{i2\pi\rho}\rangle$ be a CFNN, then the score function $Score(A)$ is defined by

$$Score(A)=(r^3-t^3-k^3)+\frac{1}{4\pi^2}(\omega^3-\theta^3-\rho^3)$$

where $Score(A)\in[-3,3]$.

Definition 4.6: Let $A=\langle r\cdot e^{i2\pi\omega},t\cdot e^{i2\pi\theta},k\cdot e^{i2\pi\rho}\rangle$ be a CFNN, then the accuracy function $Accur(A)$ is defined as follows:

$$Accur(A)=(r^3+t^3+k^3)+\frac{1}{4\pi^2}(\omega^3+\theta^3+\rho^3)$$

where $Accur(A) \in [0,3]$.

Any two CFNNs are ranked using the following method:

Let $A = \langle r_1 \cdot e^{i2\pi\omega_1}, t_1 \cdot e^{i2\pi\theta_1}, k_1 \cdot e^{i2\pi\rho_1} \rangle$ and $B = \langle r_2 \cdot e^{i2\pi\omega_2}, t_2 \cdot e^{i2\pi\theta_2}, k_2 \cdot e^{i2\pi\rho_2} \rangle$ be two CFNNs, then

- If $Score(A) > Score(B)$ then $A > B$;
- If $Score(A) = Score(B)$ then
  - If $Accur(A) < Accur(B)$ then $A < B$;
  - If $Accur(A) > Accur(B)$ then $A > B$;
  - If $Accur(A) = Accur(B)$ then $A = B$;

4.1. Decision making approach

We developed a decision making approach under CFN environment using the proposed CFNWA, CFNOWA, CFNWG, CFNOWG operators. Let $A = \{A_1, A_2, \cdots, A_m\}$ and $C = \{C_1, C_2, \cdots, C_n\}$ be the number of alternatives and criteria respectively and $w = (w_1, w_2, \cdots, w_n)^T$ where $\sum_{i=1}^{n} w_i = 1$. So, the decision matrix $A = [X_{ij}]_{m \times n}$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ is constructed where $X_{ij} = \langle r \cdot e^{i2\pi\omega}, t \cdot e^{i2\pi\theta}, k \cdot e^{i2\pi\rho} \rangle$.

Step 1: The decision matrix $A = [X_{ij}]_{m \times n}$ is transformed to a normalized matrix $B = [Y_{ij}]_{m \times n}$. For benefit and cost type of attributes the following normalization are used.

$$R_{ij} = \begin{cases} \langle r \cdot e^{i2\pi\omega}, t \cdot e^{i2\pi\theta}, k \cdot e^{i2\pi\rho} \rangle & for\ \ benefit \\ \langle k \cdot e^{i2\pi\rho}, t \cdot e^{i2\pi\theta}, r \cdot e^{i2\pi\omega} \rangle & for\ \ \cos t \end{cases}$$

Step 2: The alternatives are aggregated using the proposed operators CFNWA, CFNOWA, CFNWG, CFNOWG.

Step 3: The aggregated value's score function is calculated.

Step 4: The best alternative is selected based on ranking given in step 3.

4.2. Illustrative example

To provide evidence that the suggested arithmetic and geometric aggregation operator of CFNS, an illustrative example is discussed below. Teachers play a major role in imparting quality education to students in institutions. In this section, a decision-making approach to evaluate lecturer's research productivity at the end of year (Nguyen et al., 2020) is discussed. Assume that there are 3 lecturers to be evaluated. A committee of 4 decision makers are considered to make their assessment individually, according to their preferences of criteria and the ratings of alternative s. After a discussion with committee members five criteria are selected as in Table 18.

Table 18. Criteria for research productivity

	Criteria	Type
C_1	Number of research articles published	Benefit
C_2	Quality of journals published	Benefit
C_3	Number of books published	Benefit
C_4	Guiding postgraduate and Ph d students	Benefit
C_5	Project research grants secured	Benefit

The faculty members $A_m\ (m=1,2,3)$ are to be evaluated using CFNS by the expert under the set of 5 parameters $C_n\ (m=1,2,3,4,5)$ whose associating weighting vector is $w=(0.2,0.2,0.2,0.2,0.2)$. To select the most desirable teacher, we utilize the above step-wise procedure using CFNWA, CFNOWA, CFNWG, CFNOWG. Consider the CFNN decision matrix as stated in Table 19, Table 20, Table21 respectively.

Table 19. CFNN decision matrix for A_1

Criteria	Alternative A_1
C_1	$\langle 0.3e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.3)}, 0.5^{i2\pi(0.4)} \rangle$
C_2	$\langle 0.5e^{i2\pi(0.1)}, 0.6e^{i2\pi(0.4)}, 0.4^{i2\pi(0.3)} \rangle$
C_3	$\langle 0.4e^{i2\pi(0.3)}, 0.7e^{i2\pi(0.1)}, 0.5^{i2\pi(0.6)} \rangle$
C_4	$\langle 0.6e^{i2\pi(0.2)}, 0.5e^{i2\pi(0.2)}, 0.3^{i2\pi(0.5)} \rangle$
C_5	$\langle 0.7e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.3)}, 0.6^{i2\pi(0.3)} \rangle$

Table 20. CFNN decision matrix for A_2

Criteria	Alternative A_2
C_1	$\langle 0.6e^{i2\pi(0.1)}, 0.5e^{i2\pi(0.3)}, 0.4^{i2\pi(0.6)} \rangle$
C_2	$\langle 0.3e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.2)}, 0.7^{i2\pi(0.4)} \rangle$
C_3	$\langle 0.7e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.6)}, 0.5^{i2\pi(0.7)} \rangle$
C_4	$\langle 0.5e^{i2\pi(0.1)}, 0.6e^{i2\pi(0.6)}, 0.3^{i2\pi(0.4)} \rangle$
C_5	$\langle 0.4e^{i2\pi(0.2)}, 0.5e^{i2\pi(0.1)}, 0.3^{i2\pi(0.3)} \rangle$

Table 21. CFNN decision matrix for A_3

Criteria	Alternative A_3
C_1	$\langle 0.7e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.2)}, 0.6^{i2\pi(0.4)} \rangle$
C_2	$\langle 0.4e^{i2\pi(0.3)}, 0.5e^{i2\pi(0.4)}, 0.7^{i2\pi(0.5)} \rangle$
C_3	$\langle 0.3e^{i2\pi(0.4)}, 0.6e^{i2\pi(0.5)}, 0.4^{i2\pi(0.3)} \rangle$
C_4	$\langle 0.2e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.1)}, 0.5^{i2\pi(0.6)} \rangle$
C_5	$\langle 0.1e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.3)}, 0.3^{i2\pi(0.7)} \rangle$

Now we aggregate the criteria using the proposed CFNWA, CFNOWA, CFNWG, CFNOWG and the values are stated in Table 22, Table 23, Table 24.

Table 22. Aggregated values of A_1

Aggregation operator	A_1
CFNWA	$\left\langle 0.05e^{i2\pi(0.01)}, 0.51e^{i2\pi(0.37)}, 0.45^{i2\pi(0.64)} \right\rangle$
CFNOWA	$\left\langle 0.05e^{i2\pi(0.01)}, 0.51e^{i2\pi(0.46)}, 0.45^{i2\pi(0.63)} \right\rangle$
CFNWG	$\left\langle 0.48e^{i2\pi(0.26)}, 0.06e^{i2\pi(0.03)}, 0.04^{i2\pi(0.15)} \right\rangle$
CFNOWG	$\left\langle 0.48e^{i2\pi(0.26)}, 0.06e^{i2\pi(0.03)}, 0.04^{i2\pi(0.65)} \right\rangle$

Table 23. Aggregated values of A_2

Aggregation operator	A_2
CFNWA	$\left\langle 0.05e^{i2\pi(0.01)}, 0.54e^{i2\pi(0.46)}, 0.42^{i2\pi(0.72)} \right\rangle$
CFNOWA	$\left\langle 0.05e^{i2\pi(0.01)}, 0.54e^{i2\pi(0.46)}, 0.42^{i2\pi(0.72)} \right\rangle$
CFNWG	$\left\langle 0.48e^{i2\pi(0.26)}, 0.08e^{i2\pi(0.18)}, 0.04^{i2\pi(0.05)} \right\rangle$
CFNOWG	$\left\langle 0.48e^{i2\pi(0.26)}, 0.08e^{i2\pi(0.18)}, 0.04^{i2\pi(0.05)} \right\rangle$

Table 24. Aggregated values of the CFNN information

Aggregation operator	A_3
CFNWA	$\left\langle 0.03e^{i2\pi(0.03)}, 0.43e^{i2\pi(0.41)}, 0.48^{i2\pi(0.75)} \right\rangle$
CFNOWA	$\left\langle 0.03e^{i2\pi(0.03)}, 0.43e^{i2\pi(0.41)}, 0.48^{i2\pi(0.75)} \right\rangle$
CFNWG	$\left\langle 0.28e^{i2\pi(0.39)}, 0.03e^{i2\pi(0.07)}, 0.05^{i2\pi(0.48)} \right\rangle$
CFNOWG	$\left\langle 0.28e^{i2\pi(0.39)}, 0.03e^{i2\pi(0.07)}, 0.05^{i2\pi(0.48)} \right\rangle$

Using the score function of CFNN, we arrive at the ranking order and stated in Table 25.

Table 25. Ranking order

Aggregation operator	Ranking
Proposed CFNWA	$A_2 > A_1 > A_3$
Proposed CFNOWA	$A_2 > A_1 > A_3$
Proposed CFNWG	$A_2 > A_3 > A_1$
Proposed CFNOWG	$A_2 > A_3 > A_1$

Clearly, we can see that the CFNWA and CFNOWA provides same ranking and CFNWG and CFNOWG provides same ranking.

5. Conclusion

To model uncertain decision-making problem, graph theory plays a very important role. The novel idea of CFNG is introduced and investigated different properties of it in this paper. We have defined the order, size, degree and total degree of CFNG with appropriate examples. Also, the primary operation such as complement, union, join and ring sum of CFNG are proposed with appropriate examples and some of their important theorems are also described. It is shown with a comparison study how CFNG is more flexible than the CNG. Further, the regular graph under CFN environment is presented. Finally, we determined the use of CFNG in educational system involving decision-making and proposed the operators CFNWA, CFNOWA, CFNWG, and CFNOWG into the approach for selecting the best lecturer. In the future, we will introduce complex fermatean Dombi neutrosophic fuzzy graph and complex fermatean neutrosophic hypergraph with its applications.

Author Contributions: For Conceptualization, S.B and F.S; methodology, S.B., S.M, M.T., and A.B; software, S.M.; validation, S.B. and S.M.; formal analysis, S.M.; investigation, S.M and S.B.; resources, S.B., S.M., and T.W; writing—original draft preparation, S.M.; writing—review and editing, S.M.; visualization, S.B., and S.M.; supervision, S.B. All authors have read and agreed to the published version of the manuscript.

Funding*:* This research received no external funding.

Data Availability Statement*:* Not applicable

Conflicts of Interest*:* The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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