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Analysis of Resilient Effective Domination in Neutrosophic Under Graphs with an Application to Healthcare Resource Allocation

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ABSTRACT

Healthcare systems comprise interconnected operational units that function in environments characterised by uncertainty, constrained resources, equipment malfunctions, workforce shortages and emergency situations. To address these challenges, this study proposes the concept of resilient effective domination in neutrosophic under graphs as a framework for identifying resource-support structures capable of maintaining their effectiveness despite vertex failures. A k-resilient effective edge is defined as an edge that retains its effectiveness following the failure of up to k admissible vertices. Correspondingly, a k-resilient effective dominating set ensures that the surviving network remains effectively dominated under all possible failure configurations involving no more than k vertices. The theoretical foundations of the proposed framework are developed through the establishment of fundamental properties, encompassing monotonicity, bounds on the resilient effective domination number, vertex-deletion properties and characterisations formulated in terms of surviving effective neighbours. Furthermore, structural results are derived for complete, star, path, cycle and complete bipartite neutrosophic under graphs, demonstrating the relationship between resilience and redundancy in effective-neighbour structures. The applicability of the proposed approach is subsequently illustrated through a numerical application involving healthcare resource allocation. Neutrosophic relationship information is incorporated to identify effective relationships and facilitate a comparative assessment of ordinary and resilient effective domination. Collectively, these findings establish a quantitative framework for integrating uncertainty and tolerance to network failures into healthcare resource-allocation decisions.

1. Introduction

Real-world networks frequently function in uncertain environments, where the operational

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status and reliability of individual components may fluctuate over time. Within healthcare systems, for instance, departments and service units can experience temporary disruptions arising from equipment breakdowns, workforce shortages, scheduled maintenance, emergency shutdowns or unexpected surges in patient demand. Consequently, a resource-support arrangement that performs adequately under routine operating conditions may become insufficient when one or more constituent units are rendered unavailable. Such circumstances highlight the need to investigate domination structures that consider not only the provision of resource coverage but also the ability to sustain that coverage in the presence of operational failures.

Neutrosophic under graphs offer a mathematical approach for modelling networks whose interconnections are described through degrees of truth, indeterminacy and falsity. In this framework, effective domination is concerned with identifying vertices that provide support to other vertices through effective edges. Nevertheless, conventional effective domination principally characterises the network under its existing configuration, without explicitly evaluating whether the established dominating structure can preserve its effectiveness following vertex failures. Extending the concept of effective domination to incorporate failure tolerance is therefore essential for examining uncertain networks in which the uninterrupted provision of support must be maintained.

1.1 Literature Survey

Kumar [9] examined the complement of fuzzy graphs, contributing to the development of the structural theory of fuzzy graph representations. Their investigation offers a mathematical basis for understanding graph operations and inter-vertex relationships in environments characterised by uncertainty. Subsequently, Fei [6] explored neutrosophic graphs and their applicability to wireless communication networks, demonstrating the capacity of neutrosophic information to represent uncertainty associated with network connections. This work establishes a theoretical basis for employing neutrosophic graph structures in the analysis of resource-support networks. Similarly, Shao et al. [15] developed concepts within intuitionistic fuzzy graph theory and demonstrated their applicability to water supplier systems. Their findings illustrate how uncertain interrelationships can be modelled within supplier networks, establishing a connection between fuzzy graph theory and resource-support applications.

Shi and Kosari [16] examined domination properties within product vague graphs and demonstrated their relevance through an application in medicine. Their contribution extends the mathematical treatment of domination to product vague graph structures, providing a theoretical foundation for investigating healthcare networks. In a related development, Uluçay [19] investigated Q-neutrosophic soft graphs, focusing on their applications in operations management and communication networks. By integrating neutrosophic uncertainty with soft-set-based graph representations, this study offers an alternative mathematical framework for modelling uncertain interrelationships in interconnected systems. Alenizi [4] subsequently explored the application of neutrosophic intelligence to patient-data protection in blockchain-based smart healthcare systems. Their investigation highlights the potential of neutrosophic methodologies in addressing data-security challenges within interconnected healthcare environments.

Shambayati et al. [14] formulated a neutrosophic optimisation model addressing supply-chain virtualisation in the context of the circular economy. By employing NSGA-II, their approach addresses optimisation problems under uncertain conditions and illustrates the integration of neutrosophic modelling with resource-management optimisation. Abbas et al. [2] developed a neutrosophic Z-number-based CRITIC-EDAS model for assessing smart solar panels, combining uncertainty representation with multi-criteria decision-making techniques. Their contribution demonstrates the

applicability of neutrosophic approaches to sustainable energy-planning problems. In another related investigation, A. Fahmi [1] examined domination properties in rough m-polar fuzzy digraphs and established an application in trade networking. Their work broadens domination theory to encompass directed graph structures characterised by multiple membership values and rough uncertainty, thereby providing a related mathematical perspective for analysing uncertain network relationships.

Hu et al. [8] introduced an interval-based forecasting framework for infectious diseases through the integration of fuzzy information granulation and a spatial-temporal graph neural network. Their model incorporates uncertainty in epidemiological data while identifying spatial and temporal patterns associated with disease transmission. By generating prediction intervals instead of relying exclusively on point estimates, the proposed framework enhances the reliability of infectious disease forecasting. Sankar et al. [13] investigated network stability in signed fuzzy graphs by examining symmetric structures and domination integrity. Their research explores how underlying graph configurations influence network robustness, offering a theoretical basis for investigating resilient domination in uncertain network environments. More recently, Venkatesan et al. [20] examined balanced Fermatean quadripartitioned neutrosophic fuzzy graphs and their applications. Their contribution presents a graph-theoretic representation incorporating multiple components of uncertainty, thereby enriching the theoretical foundations available for the further development of neutrosophic graph models.

Sreelakshmi and Uma Samundesvari [17] introduced an energy-aware equitable total domination framework in neutrosophic graphs, specifically addressing satellite cluster coordination. Their approach integrates neutrosophic graph representations with domination principles and energy-conscious network management, illustrating the applicability of domination-based techniques to coordination problems in interconnected systems. Mujeeburahman et al. [12] investigated advanced domination concepts in product bipolar fuzzy graphs, developing associated theoretical properties and examining their operations and applications. This research provides complementary mathematical foundations for extending domination theory to more sophisticated fuzzy graph structures. Montenegro-Dos Santos and Zolfani [11] proposed a multi-valued neutrosophic ARAS framework to assess economic resilience in olive farming systems. Through the incorporation of neutrosophic uncertainty into alternative-ranking procedures, their approach demonstrates the potential of neutrosophic decision-making methodologies for evaluating resilience.

AbdelAziz et al. [3] introduced an integrated framework combining GIS, IoT and MCDM techniques for disaster risk management. Their methodology brings together spatial information, connected sensing technologies and multi-criteria decision-making, making the study relevant to resource-allocation problems arising in healthcare systems under uncertain operational conditions. Ali et al. [5] developed an integrated Fermatean fuzzy and neutrosophic goal-programming framework for healthcare optimisation in uncertain environments. Their model addresses multi-objective decision-making in the allocation of healthcare resources, providing a relevant contribution to the development of uncertainty-aware healthcare network models. Similarly, Sudha et al. [18] investigated fuzzy graph theory alongside neuromorphic graph models in the context of uncertain mathematical systems. By relating fuzzy graph representations to computational approaches for handling uncertainty, their research offers an additional perspective on modelling complex and uncertain interrelationships.

Marie et al. [10] introduced a bio-inspired neutrosophic-enzyme intelligence framework for the identification of paediatric dental diseases using multimodal clinical data. Their approach employs neutrosophic intelligence to facilitate disease detection in the presence of clinical uncertainty,

demonstrating the relevance of neutrosophic methodologies to healthcare applications. This contribution provides a complementary perspective to the healthcare resource-allocation application considered in the present study. Haynes et al. [7], in their comprehensive treatment of domination in graphs, brought together extensive theoretical developments and practical applications concerning domination parameters in graph theory. Their work establishes a general mathematical foundation for investigating effective domination and developing its resilient extensions. Collectively, these contributions demonstrate the growing significance of generalised graph-theoretic representations and uncertainty-based mathematical models in addressing resource allocation, network management and operational decision-making problems.

1.2 Research Gap and Motivation

Traditional domination-based frameworks primarily examine network structures in their existing operational states, without explicitly assessing the extent to which a designated dominating configuration can preserve its effectiveness when individual vertices become unavailable. In practical network environments, particularly healthcare systems, operational units may undergo temporary disruptions resulting from equipment malfunctions, personnel shortages, increased patient demand, maintenance requirements, emergency shutdowns or communication interruptions. Under such circumstances, a resource-support arrangement that provides adequate coverage during routine operations may lose its effectiveness when a critical network component becomes unavailable.

These considerations highlight the need to extend effective domination in neutrosophic under graphs through a resilience-oriented approach. The principal objective of the present investigation is to determine a minimum collection of vertices that can sustain effective domination despite the failure of a specified number of vertices. Within the proposed framework, a k-resilient effective edge is introduced to ensure that an effective edge retains its effectiveness under every admissible vertex-deletion configuration in which neither of its endpoints is removed. On this basis, a k-resilient effective dominating set is defined as a vertex set whose remaining elements continue to form an effective dominating set following any failure involving at most k vertices. The associated resilient effective domination number, denoted by γ_k^eff (G), consequently specifies the minimum number of vertices necessary to preserve effective domination within the prescribed failure-tolerance limit. The underlying research gap arises from the absence of an effective domination framework that explicitly incorporates vertex unavailability and evaluates the persistence of effective relationships following network failures. The proposed approach addresses this limitation by establishing a relationship between resilience and effective-neighbour redundancy, while investigating the minimum number of selected vertices required to maintain adequate network support under different failure configurations.

1.3 Contributions of the Present Study

The principal contributions of the present investigation are summarised as follows:

1. A comprehensive framework defining k-resilient effective edges and k-resilient effective dominating sets is developed within the context of neutrosophic under graphs.
2. Fundamental monotonicity properties are derived to establish how variations in the prescribed resilience level influence the resilient effective domination number.
3. Necessary and sufficient characterisations of resilient effective domination are established through an examination of the effective neighbours that remain available following vertex failures.
4. Lower and upper bounds for γkeff (G) are derived, accompanied by structural investigations of complete, star, path, cycle and complete bipartite neutrosophic under graphs.

5. The association between effective-neighbour redundancy and failure tolerance is analysed, thereby providing a quantitative perspective on resilience within the effective domination framework.
6. A numerical application involving healthcare resource allocation is presented to illustrate the differences between conventional and resilient effective domination under uncertain operational conditions and healthcare-unit failures.

1.4 Methodology

The investigation is conducted through a systematic mathematical modelling and analytical framework. Initially, the fundamental notation and preliminary concepts associated with neutrosophic under graphs and effective domination are formulated. Subsequently, the definitions of k-resilient effective edges and k-resilient effective dominating sets are established by incorporating admissible vertex-failure scenarios and evaluating the effective relationships preserved following each such failure. The resulting resilient effective domination number is then examined through the derivation of monotonicity properties, necessary and sufficient conditions, mathematical bounds and structural characterisations pertaining to selected classes of neutrosophic under graphs. In the final stage, a numerical healthcare resource-allocation scenario is developed in which neutrosophic values are employed to represent uncertain relationships between healthcare units. The strengths of effective edges are determined, and the proposed domination framework is utilised to assess the selected resource-support configuration across the specified failure scenarios. The resulting configuration is subsequently evaluated against ordinary effective domination to demonstrate the significance of effective-neighbour redundancy in sustaining network support.

2. PRELIMINARIES

Throughout this paper, let G = (V, E) represent a neutrosophic under graph (NUG), where V and E denote its vertex set and edge set, respectively. For any vertex v ∈ V, its effective neighbourhood is represented by NE(v). Given a subset D ⊆ V, the notation D − F denotes the subset of vertices in D that remain after the failure of a vertex set F ⊆ V. Furthermore, the order of the graph G is expressed as |V|.

Definition 2.1 Let G = (V, E) be a NUG, and consider an edge e = uv ∈ E. The edge e is defined as an effective edge if its strength satisfies the following condition:

$$\mathrm{str}(\mathrm{e}) > \mathrm{str}(\mathrm{P})$$

for every alternative u–v path P in G.

Definition 2.2 Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a $\mathcal{NUG}$. A nonempty set $\mathrm{D} \subseteq \mathrm{V}$ is called an effective dominating set of G if every vertex $\mathrm{v} \in \mathrm{V}\backslash\mathrm{D}$ has an effective neighbour in D. The minimum cardinality of an effective dominating set is called the effective domination number ($\mathcal{EDN}$) of G and is denoted by $\gamma_{\mathrm{E}}^{\mathrm{N}}(\mathrm{G})$.

Definition 2.3 Let G = (V, E) be a NUG, with e = uv ∈ E, and let k denote a non-negative integer. The edge e is defined as a k-resilient effective edge if it preserves its effectiveness in the graph G − F for every admissible failure set F ⊆ V\ satisfying |F| ≤ k.

Definition 2.4 Consider a NUG G = (V, E) and a subset D ⊆ V. For any non-negative integer k, D is defined as a k-resilient effective dominating set of G if, for every admissible failure set F ⊆ V satisfying |F| ≤ k, the surviving subset D − F constitutes an effective dominating set of the residual graph G − F.

Definition 2.5 Let G be a $\mathcal{NUG}$ that admits a k-resilient effective dominating set. The minimum cardinality of a k-resilient effective dominating set is called the k-resilient $\mathcal{EDN}$ of G and is denoted

by $\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G})$. Thus,

$$\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G}) = \min\{|\mathrm{D}|: \mathrm{D} \text{ is a } \mathrm{k} - \text{resilient effective dominating set of } \mathrm{G}\}.$$

Definition 2.6 A NUG G = (V, E) is said to satisfy the effective-edge preservation condition up to resilience level k if every effective edge with both endpoints remaining intact retains its effectiveness in the residual graph G − F for every admissible failure set F ⊆ V satisfying |F| ≤ k.

Proposition 2.1 For a NUG G, if D constitutes a k-resilient effective dominating set, then it also satisfies the requirements of an r-resilient effective dominating set for every integer r such that 0 ≤ r ≤ k.

Proposition 2.2 Let G be a NUG and suppose that D is a k-resilient effective dominating set of G. Then, for every vertex v ∈ V\D and every admissible failure set F ⊆ V satisfying |F| ≤ k and v ∉ F, the vertex v possesses at least one effective neighbour in the surviving set D − F within the residual graph G − F.

Proposition 2.3 Let G be a $\mathcal{NUG}$ and let D be a k-resilient effective dominating set. Then $\gamma_{\mathrm{E}}^{\mathrm{N}}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G})$

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Definition 3.7 Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a $\mathcal{NUG}$ and let $\mathrm{e} = \mathrm{uv} \in \mathrm{E}$. The edge e is called a k-resilient effective edge if, for every set $\mathrm{F} \subseteq \mathrm{V}\backslash\{\mathrm{u}, \mathrm{v}\}$, $|\mathrm{F}| \leq \mathrm{k}$, the edge e remains effective in the vertex-deleted $\mathrm{G} - \mathrm{F}$.

Definition 3.8 Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a $\mathcal{NUG}$ and let $\mathrm{D} \subseteq \mathrm{V}$ be an effective dominating set of G. The set D is called a k-resilient effective dominating set if, for every set $\mathrm{F} \subseteq \mathrm{V}$, $|\mathrm{F}| \leq \mathrm{k}$, the set $\mathrm{D} - \mathrm{F}$ is an effective dominating set of $\mathrm{G} - \mathrm{F}$.

Definition 3.9 9 Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a $\mathcal{NUG}$. A set $\mathrm{D} \subseteq \mathrm{V}$ is called a k-resilient effective dominating set if, for every $\mathrm{F} \subseteq \mathrm{V}$, $|\mathrm{F}| \leq \mathrm{k}$, and every vertex $\mathrm{v} \in \mathrm{V}\backslash(\mathrm{D} \cup \mathrm{F})$,there exists a vertex $\mathrm{u} \in \mathrm{D}\backslash\mathrm{F}$ such that uv is an effective edge in $\mathrm{G} - \mathrm{F}$.

Definition 3.10 Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a $\mathcal{NUG}$. The k-resilient $\mathcal{EDN}$ of G, denoted by $\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G})$, is defined as

$$\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G}) = \min\{|\mathrm{D}|: \mathrm{D} \subseteq \mathrm{V}, \quad \mathrm{D} \text{ is a } \mathrm{k} - \text{resilient effective dominating set of } \mathrm{G}\}.$$

A k-resilient effective dominating set having cardinality $\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G})$ is called a $\gamma_{\mathrm{RE}}^{\mathrm{N,k}}$-set.

Definition 3.11 A 1-resilient effective dominating set is called a single-failure resilient effective dominating set. The corresponding domination number is denoted by $\gamma_{\mathrm{RE}}^{\mathrm{N,1}}(\mathrm{G})$.

Proposition 3.4 Let G be a $\mathcal{NUG}$. If an edge $\mathrm{e} \in \mathrm{E}$ is a $(\mathrm{k}+1)$-resilient effective edge, then e is a k-resilient effective edge.

Proof. Suppose that $\mathrm{e} = \mathrm{uv}$ is a $(\mathrm{k}+1)$-resilient effective edge of G. By definition, for every set $\mathrm{F} \subseteq \mathrm{V}\backslash\{\mathrm{u}, \mathrm{v}\}$ satisfying $|\mathrm{F}| \leq \mathrm{k}+1$, the edge e remains effective in the vertex-deleted $\mathrm{G} - \mathrm{F}$.
Now let $\mathrm{F} \subseteq \mathrm{V}\backslash\{\mathrm{u}, \mathrm{v}\}$ be any set satisfying

$$|\mathrm{F}| \leq \mathrm{k}.$$

Since

$$|\mathrm{F}| \leq \mathrm{k} < \mathrm{k}+1,$$

we also have

$$|\mathrm{F}| \leq \mathrm{k}+1.$$

Therefore, by the assumption that e is a $(k+1)$-resilient effective edge, the edge e remains effective in $G-F$.
Since F was chosen arbitrarily subject to

$$F \subseteq V\backslash\{u,v\}, \qquad |F| \leq k,$$

the edge e remains effective after every admissible deletion of at most k vertices distinct from its endpoints. Hence, by the definition of k-resilient effective edge, e is a k-resilient effective edge.

Proposition 3.5 Let G be a $\mathcal{NUG}$. If D is a $(k+1)$-resilient effective dominating set of G, then D is a k-resilient effective dominating set of G.

Proof. Suppose that D is a $(k+1)$-resilient effective dominating set of G. By the definition of $(k+1)$-resilience, for every set

$$F \subseteq V, \qquad |F| \leq k+1,$$

the set

$$D-F$$

is an effective dominating set of the vertex-deleted $\mathcal{NUG}$ $G-F$.
Now let $F \subseteq V$ be any set satisfying

$$|F| \leq k.$$

Since

$$|F| \leq k < k+1,$$

we have

$$|F| \leq k+1.$$

Therefore, by the assumption that D is a $(k+1)$-resilient effective dominating set, the set

$$D-F$$

is an effective dominating set of $G-F$.
Since F was chosen arbitrarily among all subsets of V satisfying

$$|F| \leq k,$$

it follows that $D-F$ is an effective dominating set of $G-F$ for every admissible failure set F of cardinality at most k. Hence, by the definition of k-resilient effective domination, D is a k-resilient effective dominating set of G.

Proposition 3.6 Let G be a $\mathcal{NUG}$. Every k-resilient effective dominating set of G is an effective dominating set of G.

Proof. Let D be a k-resilient effective dominating set of G. By the definition of k-resilient effective domination, for every set

$$F \subseteq V, \qquad |F| \leq k,$$

the set

$$D-F$$

is an effective dominating set of the vertex-deleted $\mathcal{NUG}$ $G-F$.
Consider the particular failure set

$$F = \varnothing.$$

Then

$$|F| = 0 \leq k$$

and

$$G - F = G, \qquad D - F = D.$$

Therefore, by the definition of k-resilience,

$$D - F = D$$

is an effective dominating set of

$$G - F = G.$$

Hence, D itself is an effective dominating set of G.

Proposition 3.7 Let G be a . Suppose that no new effective neighbor of a surviving vertex is created after any single-vertex deletion. If D is a 1-resilient effective dominating set of G, then every vertex $v \in V\backslash D$ has at least two distinct effective neighbors in D.

Proof. Let D be a 1-resilient effective dominating set of G, and let

$$v \in V\backslash D.$$

Since D is an effective dominating set of G, there exists at least one vertex

$$u \in D$$

such that uv is an effective edge in G.

We show that v must have at least two distinct effective dominators in D. Suppose, to the contrary, that v has exactly one effective dominator in D, say u. Thus,

$$N_E(v) \cap D = \{u\},$$

where $N_E(v)$ denotes the set of effective neighbors of v.

Consider the failure set

$$F = \{u\}.$$

Clearly,

$$F \subseteq V, \qquad |F| = 1 \leq 1.$$

Since D is 1-resilient, the set

$$D - F = D\backslash\{u\}$$

must be an effective dominating set of the vertex-deleted $\mathcal{NUG}$

$$G - F = G - u.$$

However, $v \notin D$ and $v \neq u$, so

$$v \in V(G - u)\backslash(D - \{u\}).$$

By the assumption that no new effective neighbour of a surviving vertex is created after a single-vertex deletion, and since u is the unique effective neighbor of v in D, there is no vertex $w \in D - \{u\}$ such that wv is an effective edge in $G - u$.

This contradicts the assumption that $D - \{u\}$ is an effective dominating set of $G - u$.

Therefore, every vertex

$$v \in V\backslash D$$

must have at least two distinct effective dominators in D.

Theorem 3.1 Let G be a $\mathcal{NUG}$ and let $D \subseteq V$. Then D is a k-resilient effective dominating set if and only if, for every

$$F \subseteq V, \qquad |F| \leq k,$$

and every

$$v \in V \backslash (D \cup F),$$

there exists

$$u \in D \backslash F$$

such that uv is an effective edge in $G - F$.

Proof. ($\Rightarrow$) Suppose that D is a k-resilient effective dominating set of G. Let

$$F \subseteq V, \qquad |F| \leq k,$$

and let

$$v \in V \backslash (D \cup F).$$

Since D is k-resilient, by definition,

$$D - F$$

is an effective dominating set of the vertex-deleted $\mathcal{NUG}$ $G - F$.

Since

$$v \in V \backslash (D \cup F),$$

the vertex v belongs to $G - F$ and does not belong to $D - F$. Therefore, there exists a vertex

$$u \in D - F$$

such that uv is an effective edge in $G - F$.

Hence, for every admissible failure set F and every vertex

$$v \in V \backslash (D \cup F),$$

there exists

$$u \in D \backslash F$$

such that uv is an effective edge in $G - F$.

($\Leftarrow$) Conversely, suppose that for every

$$F \subseteq V, \qquad |F| \leq k,$$

and every

$$v \in V \backslash (D \cup F),$$

there exists a vertex

$$u \in D \backslash F$$

such that uv is an effective edge in $G - F$.

Let $F \subseteq V$ be arbitrary with

$$|F| \leq k.$$

We need to show that $D - F$ is an effective dominating set of $G - F$.

Consider any vertex

$$v \in V(G - F) \backslash (D - F).$$

Since

$$V(G - F) = V \backslash F$$

and

$$D - F = D \backslash F,$$

we have

$$v \in (V \backslash F) \backslash (D \backslash F) = V \backslash (D \cup F).$$

By the assumed condition, there exists

$$u \in D \backslash F = D - F$$

such that uv is an effective edge in $G - F$.

Thus, every vertex of $G - F$ that does not belong to $D - F$ is effectively dominated by a vertex of $D - F$. Therefore,

$$D - F$$

is an effective dominating set of $G - F$.
Since F was arbitrary subject to

$$|F| \leq k,$$

it follows from the definition of k-resilient effective domination that D is a k-resilient effective dominating set of G.
Hence,

$$D \text{ is a } k - \text{resilient effective dominating set}$$

if and only if the stated vertex-wise effective domination condition holds.

Theorem 3.2 Let G be a NUG in which the corresponding resilient effective dominating sets are well-defined and exist. Under these conditions, the following inequality is satisfied:

$$\gamma_E^N(G) \leq \gamma_{RE}^{(N,1)}(G) \leq \gamma_{RE}^{(N,2)}(G) \leq \cdots \leq \gamma_{RE}^{(N,k)}(G) \leq \gamma_{RE}^{(N,k+1)}(G).$$

Proof. For each positive integer k, denote by

$D_{RE}^k(G) = \{D \subseteq V: D \text{ is a } k\text{-resilient effective dominating set of } G\}$ the collection of all k-resilient effective dominating sets of G.

The preceding proposition establishes that any (k+1)-resilient effective dominating set necessarily satisfies the requirements of a k-resilient effective dominating set. Consequently, the associated set families are related by

$$D_{RE}^{(k+1)}(G) \subseteq D_{RE}^k(G).$$

This inclusion property extends to successive resilience levels. In particular, for every integer r such that 1≤r≤k, we obtain

$$D_{RE}^{(r+1)}(G) \subseteq D_{RE}^r(G).$$

Since all the relevant families are non-empty by assumption, the minimum cardinality associated with a larger resilience requirement cannot be smaller than that corresponding to the

preceding level. Hence,

$$\gamma_{\mathrm{RE}}^{(\mathrm{N},\mathrm{r})}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{(\mathrm{N},\mathrm{r}+1)}(\mathrm{G}),\ 1 \leq \mathrm{r} \leq \mathrm{k}.$$

Repeated application of this relation produces the chain

$$\gamma_{\mathrm{RE}}^{(\mathrm{N},1)}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{(\mathrm{N},2)}(\mathrm{G}) \leq \cdots \leq \gamma_{\mathrm{RE}}^{(\mathrm{N},\mathrm{k})}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{(\mathrm{N},\mathrm{k}+1)}(\mathrm{G}).$$

Moreover, as demonstrated in the preceding proposition, the 1-resilient effective domination property is sufficient to ensure effective domination. Thus, every member of DRE1(G) also belongs to the family of effective dominating sets, giving

$$\mathrm{D}_{\mathrm{RE}}^{1}(\mathrm{G}) \subseteq \mathrm{D}_{\mathrm{E}}(\mathrm{G}),$$

where DE(G) denotes the collection of all effective dominating sets of G. Therefore, comparison of the minimum cardinalities of these families' yields

$\gamma_{\mathrm{E}}^{\mathrm{N}}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{(\mathrm{N},1)}(\mathrm{G})$.Combining this result with the established monotonicity of the resilient effective domination numbers, we arrive at

$$\gamma_{\mathrm{E}}^{\mathrm{N}}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{(\mathrm{N},1)}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{(\mathrm{N},2)}(\mathrm{G}) \leq \cdots \leq \gamma_{\mathrm{RE}}^{(\mathrm{N},\mathrm{k})}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{(\mathrm{N},\mathrm{k}+1)}(\mathrm{G}).$$

Hence, the asserted inequality is proved.

Theorem 3.3 Let G be a complete $\mathcal{NUG}$ in which every pair of distinct vertices is joined by an effective edge. Suppose, in addition, that the effective-edge property is preserved after the deletion of any set of at most k vertices. If

$$|\mathrm{V}| \geq \mathrm{k}+2,$$

then

$$\gamma_{\mathrm{RE}}^{\mathrm{N},\mathrm{k}}(\mathrm{G}) = \mathrm{k}+1.$$

Proof. Let

$$\mathrm{n} = |\mathrm{V}|.$$

We prove the result by establishing the upper and lower bounds separately.
First, choose any subset

$$\mathrm{D} \subseteq \mathrm{V}$$

such that

$$|\mathrm{D}| = \mathrm{k}+1.$$

We claim that D is a k-resilient effective dominating set of G.
Let

$$\mathrm{F} \subseteq \mathrm{V}, \qquad |\mathrm{F}| \leq \mathrm{k}.$$

Consider the vertex-deleted graph $\mathrm{G}-\mathrm{F}$. Since

$$|\mathrm{D}| = \mathrm{k}+1 > |\mathrm{F}|,$$

we have

$$\mathrm{D}-\mathrm{F} \neq \varnothing.$$

Moreover, since

$$|V| \geq k + 2 > |F|,$$

the graph $G - F$ contains at least two vertices.
Let

$$v \in V \backslash (D \cup F).$$

Since G is complete, v is adjacent to every vertex of $D - F$. By the assumed preservation of the effective-edge property, every such edge remains effective in $G - F$. Since

$$D - F \neq \varnothing,$$

there exists some

$$u \in D - F$$

such that uv is an effective edge in $G - F$.
Therefore, every vertex

$$v \in V \backslash (D \cup F)$$

is effectively dominated by a vertex of $D - F$ in $G - F$. By the characterization of k-resilient effective domination, D is a k-resilient effective dominating set of G.
Hence,

$$\gamma_{RE}^{N,k}(G) \leq k + 1.$$

Next, we prove the reverse inequality. Suppose, to the contrary, that there exists a k-resilient effective dominating set D satisfying

$$|D| \leq k.$$

Choose

$$F = D.$$

Then

$$|F| = |D| \leq k.$$

Thus F is an admissible failure set.
After deleting $F = D$, we have

$$D - F = D - D = \varnothing.$$

On the other hand, since

$$|V| \geq k + 2$$

and

$$|F| \leq k,$$

it follows that

$$|V \backslash F| \geq k + 2 - k = 2.$$

Consequently, $G - F$ contains at least one vertex outside $D - F = \varnothing$.
Such a vertex cannot be effectively dominated by $D - F$, because

$$D - F = \varnothing.$$

This contradicts the assumption that D is a k-resilient effective dominating set.

Therefore, every k-resilient effective dominating set D satisfies

$$|D| \geq k + 1.$$

Hence,

$$\gamma_{RE}^{N,k}(G) \geq k + 1.$$

Combining the two bounds,

$$\gamma_{RE}^{N,k}(G) \leq k + 1$$

and

$$\gamma_{RE}^{N,k}(G) \geq k + 1,$$

we obtain

$$\gamma_{RE}^{N,k}(G) = k + 1.$$

This completes the proof.

Proposition 3.8 Let P_n be a neutrosophic under path in which every edge is effective. Then every edge of P_n is a 0-resilient effective edge.

Proof. Let

$$P_n = v_1 v_2 \cdots v_n$$

be a neutrosophic under path in which every edge is effective. Let

$$e = v_i v_{i+1}$$

be an arbitrary edge of P_n, where

$$1 \leq i \leq n - 1.$$

By the definition of a 0-resilient effective edge, we consider every set

$$F \subseteq V(P_n) \backslash \{v_i, v_{i+1}\}$$

satisfying

$$|F| \leq 0.$$

The only such set is

$$F = \varnothing.$$

Hence,

$$P_n - F = P_n.$$

By the hypothesis, every edge of P_n is effective. In particular,

$$e = v_i v_{i+1}$$

is an effective edge in $P_n - F = P_n$.

Therefore, e remains effective for every admissible failure set F satisfying

$$|F| \leq 0.$$

Thus, by the definition of a 0-resilient effective edge, e is a 0-resilient effective edge.

Since e was chosen arbitrarily, every edge of P_n is a 0-resilient effective edge.

Proposition 3.9 Let $P_n = v_1 v_2 \cdots v_n$ be a neutrosophic under path with $n \geq 3$ and let

$$e = v_i v_{i+1}$$

be an internal edge, where

$$2 \leq i \leq n - 2.$$

Then, for every vertex

$$w \in V(P_n) \backslash \{v_i, v_{i+1}\},$$

the edge e remains present in the vertex-deleted $\mathcal{NUG}$

$$P_n - w.$$

Proof. Let

$$e = v_i v_{i+1}$$

be an internal edge of P_n, where

$$2 \leq i \leq n - 2.$$

Let

$$w \in V(P_n) \backslash \{v_i, v_{i+1}\}$$

be arbitrary.

Since w is distinct from both v_i and v_{i+1}, deleting the vertex w removes only the vertex w and all edges incident with w. In particular, neither endpoint of the edge

$$e = v_i v_{i+1}$$

is deleted.
Therefore,

$$v_i, v_{i+1} \in V(P_n - w),$$

and the edge joining these two consecutive vertices remains present in the resulting vertex-deleted $\mathcal{NUG}$. Hence,

$$e = v_i v_{i+1} \in E(P_n - w).$$

Since w was chosen arbitrarily from

$$V(P_n) \backslash \{v_i, v_{i+1}\},$$

the edge e remains present in $P_n - w$ for every such vertex deletion.

Therefore, the edge e remains present in $P_n - w$ after the deletion of any vertex other than its endpoints.

Theorem 3.4 Let C_n be a neutrosophic under cycle and let $e = uv$ be an edge of C_n. If the strength of an alternative u–v path in C_n is greater than or equal to the strength of e, then e is not an effective edge. Consequently, if every edge of C_n satisfies this condition, then C_n admits no effective dominating set based on effective edges.
Proof. Let

$$C_n = v_1 v_2 \cdots v_n v_1$$

be a neutrosophic under cycle and let

$$e = uv$$

be an arbitrary edge of C_n.

Since C_n is a cycle, there exists an alternative u–v path P in C_n that does not contain the edge e. By hypothesis, the strength of this alternative path satisfies

$$str(P) \geq str(e).$$

Recall that an edge is effective only when its strength is strictly greater than the strength of every alternative path between its endpoints. In particular, for e to be an effective edge, it is necessary that

$$str(e) > str(P).$$

However, the assumed inequality

$$str(P) \geq str(e)$$

implies

$$str(e) \leq str(P),$$

which contradicts the necessary condition for e to be effective. Therefore, e is not an effective edge.

Now suppose that every edge of C_n has an alternative path between its endpoints whose strength is greater than or equal to the strength of that edge. Applying the above argument to every edge of C_n, we conclude that no edge of C_n is effective. An effective dominating set based on effective edges requires every vertex outside the dominating set to be effectively dominated by an effective edge. Since C_n contains no effective edges, no vertex can be effectively dominated through an effective edge. Hence, C_n admits no effective dominating set based on effective edges.

Proposition 3. 10 Let $G = K_{1,n}$ be a star $\mathcal{NUG}$ in which every center–leaf edge is effective. Then

$$\{c\}$$

is an effective dominating set, where c is the center of the star.

Proof. Let

$$G = K_{1,n}$$

be a star $\mathcal{NUG}$ with center c and leaves

$$v_1, v_2, \dots, v_n.$$

By hypothesis, every centre–leaf edge

$$cv_i, \qquad 1 \leq i \leq n,$$

is an effective edge.

Consider the set

$$D = \{c\}.$$

For every leaf v_i of G, we have

$$v_i \notin D$$

and

$$cv_i \in E.$$

Moreover, by hypothesis, cv_i is an effective edge. Hence, every vertex of

$$V \backslash D = \{v_1, v_2, \dots, v_n\}$$

is effectively dominated by the centre $c \in D$.

Therefore, every vertex outside D has an effective neighbor in D. By the definition of effective domination, $D = \{c\}$ is an effective dominating set of G.
Hence,

$$\{c\}$$

is an effective dominating set of the star $\mathcal{NUG}$ G.
Theorem 3.5 Let $G = K_{1,n}$ be a star $\mathcal{NUG}$ with $n \geq 2$, and let c be its center. Then the set

$$D = \{c\}$$

is not a 1-resilient effective dominating set if resilience is required against the failure of the center vertex under the open-neighborhood interpretation of effective domination.
Proof. Let

$$D = \{c\}$$

be the effective dominating set consisting of the centre vertex of G.
Consider the failure set

$$F = \{c\}.$$

Clearly,

$$F \subseteq V, \qquad |F| = 1.$$

After the failure of the centre vertex, the resulting graph is

$$G - F = G - c,$$

which consists of the n leaves

$$v_1, v_2, \dots, v_n$$

with no edges between them.
Furthermore,

$$D - F = \{c\} - \{c\} = \varnothing.$$

Since $n \geq 2$, at least one vertex, in fact all n leaves, remains in $G - F$. Under the open-neighborhood interpretation of effective domination, a surviving vertex must have an effective neighbor belonging to the dominating set. However,

$$D - F = \varnothing,$$

so no surviving leaf has a vertex in $D - F$ that can effectively dominate it.
Thus,

$$D - F$$

is not an effective dominating set of $G - F$.
Therefore, $D = \{c\}$ fails to satisfy the defining condition for 1-resilient effective domination. Hence,

$$\{c\}$$

is not a 1-resilient effective dominating set when failure of the center vertex is allowed.

Proposition 3.11 Let $G = K_{m,n}$ be a complete bipartite $\mathcal{NUG}$ in which every edge between the two partite sets is effective. If

$$m \geq k + 1,$$

then the partite set of cardinality m is a k-resilient effective dominating set, provided that the effective-edge property is preserved after the deletion of at most k vertices.

Proof. Let the two partite sets of G be

$$X = \{x_1, x_2, \dots, x_m\} \quad \text{and} \quad Y = \{y_1, y_2, \dots, y_n\},$$

with

$$|X| = m, \qquad |Y| = n,$$

and consider

$$D = X.$$

We prove that D is a k-resilient effective dominating set.

Let

$$F \subseteq V(G), \qquad |F| \leq k$$

be an arbitrary failure set. Since $m \geq k + 1$, at most k vertices of X can be removed. Hence,

$$|X - F| \geq m - k \geq 1.$$

Thus, at least one vertex of X survives after the failure set F is removed.

Now consider any vertex

$$y \in Y \backslash F.$$

Since G is complete bipartite, y is adjacent to every vertex of X. In particular, because $X - F \neq \varnothing$, there exists a vertex

$$x \in X - F$$

such that

$$xy \in E(G - F).$$

By hypothesis, the effective-edge property is preserved after the deletion of at most k vertices. Since $|F| \leq k$ and every edge between X and Y is effective in G, the edge xy remains effective in $G - F$.

Therefore, every vertex

$$y \in Y \backslash F$$

has an effective neighbour in

$$D - F = X - F.$$

Moreover, vertices of $X - F$ do not require domination because $D - F$ is the dominating set and effective domination is interpreted through open neighborhoods of vertices outside the dominating set.

Hence,

$$D - F$$

is an effective dominating set of $G - F$ for every

$$F \subseteq V(G), \qquad |F| \leq k.$$

Therefore, $D = X$ is a k-resilient effective dominating set of G.

Theorem 3.6 Let $G = K_{m,n}$ be a complete bipartite $\mathcal{NUG}$ satisfying the effective-edge

preservation condition. If

$$m \geq k+1 \qquad \text{and} \qquad n \geq k+1,$$

then both partite sets are k-resilient effective dominating sets.
Proof. Let the two partite sets of G be

$$X = \{x_1, x_2, \dots, x_m\} \quad \text{and} \quad Y = \{y_1, y_2, \dots, y_n\},$$

with

$$|X| = m, \qquad |Y| = n.$$

First, consider $D = X$. Since

$$m \geq k+1,$$

the preceding proposition implies that X is a k-resilient effective dominating set, provided that the effective-edge property is preserved after the deletion of at most k vertices. This condition is satisfied by the hypothesis of the theorem.
Similarly, consider $D = Y$. Since

$$n \geq k+1,$$

the same proposition, with the roles of the two partite sets interchanged, implies that Y is a k-resilient effective dominating set.
More explicitly, let

$$F \subseteq V(G), \qquad |F| \leq k.$$

Because $m \geq k+1$, we have

$$|X - F| \geq m - k \geq 1.$$

Thus, for every vertex

$$y \in Y \backslash F,$$

there exists a vertex $x \in X - F$ adjacent to y. By the effective-edge preservation condition, the edge xy remains effective in $G - F$. Hence, $X - F$ effectively dominates every vertex of $Y - F$.
Likewise, because $n \geq k+1$,

$$|Y - F| \geq n - k \geq 1.$$

Therefore, for every vertex

$$x \in X \backslash F,$$

there exists a vertex $y \in Y - F$ such that xy is an effective edge in $G - F$. Hence, $Y - F$ effectively dominates every vertex of $X - F$.
Consequently, for every failure set $F \subseteq V(G)$ with $|F| \leq k$, both

$$X - F \quad \text{and} \quad Y - F$$

are effective dominating sets of $G - F$. Therefore, both partite sets X and Y are k-resilient effective dominating sets of G.

Proposition 3.12 Let $G = (V, E)$ be a $\mathcal{NUG}$ and let D be a k-resilient effective dominating set of G. Then, for every

$$F \subseteq V, \qquad |F| \leq k,$$

the set $D - F$ is an effective dominating set of $G - F$.
Proof. Since D is a k-resilient effective dominating set of G, by definition, for every failure set

$$F \subseteq V, \qquad |F| \leq k,$$

the set $D - F$ must be an effective dominating set of the resulting graph $G - F$.
More precisely, let

$$v \in V(G - F) \backslash (D - F).$$

Then

$$v \in V \backslash (D \cup F).$$

Because D is k-resilient, there exists a vertex

$$u \in D \backslash F$$

such that

$$uv \in E(G - F)$$

is an effective edge in $G - F$. Hence, every vertex of

$$V(G - F) \backslash (D - F)$$

has an effective neighbour in $D - F$.
Therefore, $D - F$ is an effective dominating set of $G - F$ for every

$$F \subseteq V, \qquad |F| \leq k.$$

Theorem 3.7 Let $G = (V, E)$ be a $\mathcal{NUG}$ and let D be a k-resilient effective dominating set of G. For every

$$F \subseteq V, \qquad |F| \leq k,$$

we have

$$\gamma_E^N(G - F) \leq |D - F|.$$

Proof. Let

$$F \subseteq V, \qquad |F| \leq k.$$

Since D is a k-resilient effective dominating set of G, by the preceding proposition,

$$D - F$$

is an effective dominating set of the resulting graph $G - F$.

By definition, $\gamma_E^N(G - F)$ denotes the minimum cardinality of an effective dominating set of $G - F$. Since $D - F$ is one such effective dominating set, its cardinality provides an upper bound for the minimum. Therefore,

$$\gamma_E^N(G - F) \leq |D - F|.$$

Hence, for every

$$F \subseteq V, \qquad |F| \leq k,$$

we have

$$\gamma_E^N(G - F) \leq |D - F|.$$

Proposition 3.13 Let $G = (V, E)$ be a $\mathcal{NUG}$ and let D be a k-resilient effective dominating set of

G. If

$$F \subseteq D, \qquad |F| \leq k,$$

then $D - F$ is an effective dominating set of $G - F$.
Proof. Since

$$F \subseteq D \subseteq V$$

and

$$|F| \leq k,$$

the set F is an admissible failure set in the definition of k-resilience.
Because D is a k-resilient effective dominating set of G, it follows that for every failure set

$$F \subseteq V, \qquad |F| \leq k,$$

the set $D - F$ is an effective dominating set of $G - F$.
Therefore, in particular, when

$$F \subseteq D, \qquad |F| \leq k,$$

we have

$$D - F$$

as an effective dominating set of $G - F$.
Hence the result follows.

Theorem 3.8 Let $G = (V, E)$ be a $\mathcal{NUG}$ of order n. If G admits a k-resilient effective dominating set, then

$$k + 1 \leq \gamma_{RE}^{N,k}(G) \leq n,$$

provided that

$$n \geq k + 2.$$

Proof. Let D be a k-resilient effective dominating set of G. We first establish the lower bound.
Suppose, to the contrary, that

$$|D| \leq k.$$

Choose the failure set

$$F = D.$$

Then

$$F \subseteq V, \qquad |F| = |D| \leq k.$$

By the k-resilience of D, the set

$$D - F = D - D = \varnothing$$

must be an effective dominating set of $G - F$.
On the other hand, since G has order n and

$$n \geq k + 2,$$

we have

$$|V \backslash F| = n - |F| \geq n - k \geq 2.$$

Thus, $G - F$ contains at least two vertices, all of which lie outside the empty set $D - F$. Under the open-neighborhood interpretation of effective domination, every vertex outside the dominating set must have an effective neighbor in the dominating set. Since

$$D - F = \varnothing,$$

no vertex of $G - F$ can be effectively dominated. This is a contradiction.
Therefore, every k-resilient effective dominating set D satisfies

$$|D| \geq k + 1.$$

Taking the minimum cardinality over all k-resilient effective dominating sets gives

$$\gamma_{RE}^{N,k}(G) \geq k + 1.$$

For the upper bound, consider

$$D = V.$$

For every failure set

$$F \subseteq V, \qquad |F| \leq k,$$

we have

$$D - F = V - F = V(G - F).$$

Thus, there are no vertices in $G - F$ outside the dominating set $D - F$. Hence, under the open-neighborhood interpretation, the effective domination condition is satisfied vacuously. Therefore, V is a k-resilient effective dominating set of G.
Consequently,

$$\gamma_{RE}^{N,k}(G) \leq |V| = n.$$

Combining the two bounds, we obtain

$$k + 1 \leq \gamma_{RE}^{N,k}(G) \leq n.$$

Proposition 3.14 Let $G = (V, E)$ be a $\mathcal{NUG}$ of order n and let D be a k-resilient effective dominating set. If

$$n > k$$

and

$$|D| \leq k,$$

then D cannot be a k-resilient effective dominating set.
Proof. Suppose, to the contrary, that D is a k-resilient effective dominating set of G and that

$$|D| \leq k.$$

Consider the failure set

$$F = D.$$

Since

$$F \subseteq V$$

and

$$|F| = |D| \leq k,$$

the set F is an admissible failure set for k-resilience.
After the failure of all vertices in D, we have

$$D - F = D - D = \varnothing.$$

On the other hand,

$$|V(G - F)| = |V| - |F| = n - |D|.$$

Since

$$n > k \geq |D|,$$

it follows that

$$n - |D| > 0.$$

Therefore,

$$V(G - F) \neq \varnothing.$$

Thus, at least one vertex survives in $G - F$, but

$$D - F = \varnothing.$$

Under the open-neighbourhood interpretation of effective domination, every vertex outside the dominating set must have an effective neighbour in the dominating set. Since the dominating set is empty, no surviving vertex can be effectively dominated. This contradicts the assumption that D is k-resilient. Hence, any k-resilient effective dominating set must satisfy

$$|D| \geq k + 1.$$

Therefore, a set D with

$$|D| \leq k$$

cannot be a k-resilient effective dominating set.

Theorem 3.9 Let G be a complete NUG in which every pair of distinct vertices is connected by an effective edge. Assume, moreover, that the effectiveness of these edges is preserved following the deletion of any subset containing at most k vertices. If

$$| V | \geq k + 2,$$

then every subset $D \subseteq V$ satisfying

$$| D | \geq k + 1$$

constitutes a k-resilient effective dominating set of G.
Proof. Consider an arbitrary subset $D \subseteq V$ such that

$$| D | \geq k + 1.$$

We establish that D satisfies the defining conditions of k-resilient effective domination. Let $F \subseteq V$ be any admissible failure set with

$$| F | \leq k.$$

Since the cardinality of D is at least k+1, the removal of at most k vertices cannot eliminate all its elements. More precisely,

$$| D - F | \geq | D | - | F | \geq k + 1 - k = 1.$$

It follows that

$$D - F \neq \emptyset.$$

Next, select an arbitrary vertex

$$v \in V(G - F) \setminus (D - F).$$

By the definition of the remaining vertex and dominating sets, we have

$$v \in V \setminus (D \cup F).$$

Since G is complete, every pair of distinct vertices is adjacent. Furthermore, the non-emptiness of D-F guarantees the existence of at least one surviving vertex u∈D-F. Clearly, u≠v, and consequently,

$$uv \in E(G).$$

By hypothesis, every edge connecting distinct vertices of G is effective. In addition, the stipulated preservation condition ensures that deleting at most k vertices does not destroy the effectiveness of any surviving edge. Since |F|≤k, the edge uv therefore remains effective in G-F. Hence, the arbitrary vertex v has an effective neighbour u belonging to the surviving dominating set D-F. This establishes that every vertex outside D-F is effectively dominated within the resulting graph. Accordingly,

$$D - F$$

is an effective dominating set of G-F.

Since the failure set F was chosen arbitrarily subject to |F|≤k, the effective domination property persists under every admissible vertex deletion. Therefore, D satisfies the defining requirements of k-resilient effective domination. Thus, every subset D⊆V with |D|≥k+1 is a k-resilient effective dominating set of G.

Proposition 3. 15 Let G be a $\mathcal{NUG}$. If G admits a k-resilient effective dominating set for some positive integer k, then

$$\gamma_{RE}^{N,r}(G) \leq \gamma_{RE}^{N,k}(G)$$

for every integer

$$0 \leq r \leq k.$$

Proof. Since G admits a k-resilient effective dominating set, the resilient $\mathcal{EDN}$

$$\gamma_{RE}^{N,k}(G)$$

is well-defined. Let D_k be a $\gamma_{RE}^{N,k}$-set of G. Thus,

$$|D_k| = \gamma_{RE}^{N,k}(G),$$

and D_k is a k-resilient effective dominating set.
Let r be any integer satisfying

$$0 \leq r \leq k.$$

By the preceding proposition, every k-resilient effective dominating set is also an r-resilient effective dominating set. Hence, D_k is an r-resilient effective dominating set of G.
Therefore, by the definition of the minimum resilient $\mathcal{EDN}$,

$$\gamma_{RE}^{N,r}(G) \leq |D_k|.$$

Since

$$|D_k| = \gamma_{RE}^{N,k}(G),$$

we obtain

$$\gamma_{RE}^{N,r}(G) \leq \gamma_{RE}^{N,k}(G).$$

Thus,

$$\gamma_{RE}^{N,r}(G) \leq \gamma_{RE}^{N,k}(G)$$

for every integer r satisfying $0 \leq r \leq k$.

Proposition 3.16 Let G be a $\mathcal{NUG}$ and let D be an effective dominating set. Suppose that, under every vertex deletion, no new effective neighbor of a surviving vertex is created. If a vertex

$$v \in V \backslash D$$

has at most k effective neighbors in D, then D is not a k-resilient effective dominating set.
Proof. Let

$$N_E(v) \cap D = \{u_1, u_2, \dots, u_t\},$$

where $N_E(v)$ denotes the set of effective neighbors of v, and suppose that

$$t \leq k.$$

Define the failure set

$$F = N_E(v) \cap D.$$

Then

$$F \subseteq D, \qquad |F| = t \leq k.$$

Since $v \notin D$ and $F \subseteq D$, we have

$$v \notin F.$$

Hence, v survives in the remaining graph $G - F$.

Moreover, by the construction of F, every effective neighbor of v that belongs to D has been removed. Thus,

$$N_E(v) \cap (D - F) = \varnothing.$$

By the assumed condition that vertex deletion does not create any new effective neighbour of a surviving vertex, v has no effective neighbor in $D - F$ in $G - F$.
Therefore, v is not effectively dominated by $D - F$ in $G - F$. Consequently,

$$D - F$$

is not an effective dominating set of $G - F$.
Since

$$F \subseteq V, \qquad |F| \leq k,$$

this contradicts the requirement for k-resilience. Hence,

$$Disnotak - resilienteffectivedominatingset.$$

Proposition 3.17 Let $P_n = v_1 v_2 \cdots v_n$ be a neutrosophic under path in which every edge is effective. If an internal vertex $v_i,\ 2 \leq i \leq n-1$, is deleted, then every edge incident with v_i is absent from $P_n - v_i$.

Proof. Let

$$P_n = v_1 v_2 \cdots v_n$$

be a neutrosophic under path. Since v_i is an internal vertex with

$$2 \leq i \leq n-1,$$

it is incident with the edges

$$v_{i-1}v_i \quad \text{and} \quad v_i v_{i+1}.$$

Consider the graph obtained by deleting v_i from P_n. By the definition of vertex deletion, the vertex v_i is removed together with all edges incident with it. Hence,

$$v_{i-1}v_i \notin E(P_n - v_i)$$

and

$$v_i v_{i+1} \notin E(P_n - v_i).$$

Therefore, every edge incident with the deleted vertex v_i is absent from the resulting graph $P_n - v_i$.

Thus, the result follows.

Proposition 3.18 Let $P_n = v_1 v_2 \cdots v_n$ be a neutrosophic under path in which every edge is effective. If

$$n \geq 3$$

and

$$e = v_i v_{i+1}, \quad 2 \leq i \leq n-2,$$

then for every

$$F \subseteq V(P_n) \backslash \{v_i, v_{i+1}\}, \quad |F| \leq 1,$$

the edge e remains present in $P_n - F$.

Proof. Let

$$e = v_i v_{i+1}$$

be an internal edge of P_n, where

$$2 \leq i \leq n-2.$$

Let

$$F \subseteq V(P_n) \backslash \{v_i, v_{i+1}\}, \quad |F| \leq 1.$$

Since neither v_i nor v_{i+1} belongs to F, both endpoints of the edge e remain in the vertex set of $P_n - F$.

Vertex deletion removes only the vertices belonging to F together with the edges incident with those deleted vertices. Since

$$v_i, v_{i+1} \notin F,$$

the edge

$$e = v_i v_{i+1}$$

is not incident with any deleted vertex. Hence,

$$e \in E(P_n - F).$$

Therefore, the edge $v_i v_{i+1}$ remains present in $P_n - F$ for every admissible failure set F with

$$|F| \leq 1.$$

Proposition 3.19 Let $P_n = v_1 v_2 \cdots v_n$ be a neutrosophic under path in which every edge is effective. If

$$e = v_i v_{i+1},$$

then deletion of any vertex different from v_i and v_{i+1} preserves the existence of the edge e in the resulting graph.

Proof. Let

$$e = v_i v_{i+1}$$

be an edge of the path P_n. Let w be any vertex of P_n such that

$$w \neq v_i \quad \text{and} \quad w \neq v_{i+1}.$$

Consider the graph obtained by deleting w, namely

$$P_n - w.$$

By the definition of vertex deletion, the vertex w and all edges incident with w are removed. Since

$$w \notin \{v_i, v_{i+1}\},$$

the edge

$$e = v_i v_{i+1}$$

is not incident with the deleted vertex w.

Therefore, the endpoints v_i and v_{i+1} remain in the resulting graph, and consequently

$$e = v_i v_{i+1} \in E(P_n - w).$$

Hence, deletion of any vertex different from v_i and v_{i+1} preserves the existence of the edge e in the resulting graph.

Theorem 3.10 Let C_n be a neutrosophic under cycle and let $e = uv$ be an edge of C_n. If the strength of the alternative u–v path is at least the strength of e, then e is not an effective edge.

Proof. Let

$$e = uv$$

be an edge of the neutrosophic under cycle C_n. Since C_n is a cycle, there exists an alternative u–v path in C_n that does not contain the edge e.

Let the strength of the edge e be denoted by

$$\mathrm{str}(e),$$

and let the strength of the alternative u–v path be denoted by

$$\mathrm{str}(P_{uv}).$$

By hypothesis,

$$\mathrm{str}(P_{uv}) \geq \mathrm{str}(e).$$

According to the definition of an effective edge, an edge $e = uv$ is effective only when the strength of the edge is strictly greater than the strength of every alternative u–v path. In particular, effectiveness requires

$$\mathrm{str}(e) > \mathrm{str}(P_{uv}).$$

However, the hypothesis gives

$$\mathrm{str}(P_{uv}) \geq \mathrm{str}(e),$$

which contradicts the strict inequality required for effectiveness.
Therefore, the edge $e = uv$ is not an effective edge.

Proposition 3.20 Let C_n be a neutrosophic under cycle and let $e = uv$ be an edge of C_n. If the strength of the unique alternative u–v path is strictly smaller than the strength of e, then e is an effective edge.
Proof. Let

$$e = uv$$

be an edge of the neutrosophic under cycle C_n. Since C_n is a cycle, there exists a unique alternative u–v path in C_n that does not contain the edge e. Denote this alternative path by P_{uv}.
Let the strength of the edge e be denoted by

$$\mathrm{str}(e),$$

and the strength of the alternative path P_{uv} be denoted by

$$\mathrm{str}(P_{uv}).$$

By hypothesis,

$$\mathrm{str}(P_{uv}) < \mathrm{str}(e).$$

Equivalently,

$$\mathrm{str}(e) > \mathrm{str}(P_{uv}).$$

By the definition of an effective edge, an edge $e = uv$ is effective when its strength is strictly greater than the strength of the alternative u–v path. Since P_{uv} is the unique alternative u–v path in C_n and

$$\mathrm{str}(e) > \mathrm{str}(P_{uv}),$$

the edge e satisfies the required condition for effectiveness.
Therefore,

$$e = uv$$

is an effective edge.

Theorem 3. 11 Let C_n be a neutrosophic under cycle. If every edge of C_n has an alternative path between its endpoints whose strength is at least the strength of that edge, then C_n has no effective edges and hence admits no effective dominating set.
Proof. Let $C_n = (V, E)$ be a neutrosophic under cycle. Consider an arbitrary edge

$$e = uv \in E.$$

By hypothesis, there exists an alternative u–v path, say P_{uv}, such that

$$\mathrm{str}(P_{uv}) \geq \mathrm{str}(e).$$

By the preceding theorem, whenever the strength of an alternative u–v path is at least the strength of the edge $e = uv$, the edge e is not effective. Hence,

$$e \notin E_{eff}.$$

Since e was chosen arbitrarily from E, it follows that no edge of C_n is effective. Therefore,

$$E_{eff} = \varnothing.$$

Now suppose, to the contrary, that C_n admits an effective dominating set D. Since C_n is a cycle, $|V(C_n)| \geq 3$. Hence, for any proper dominating set $D \subsetneq V(C_n)$, there exists a vertex

$$v \in V(C_n) \backslash D.$$

By the definition of effective domination, v must have an effective neighbor in D. This is impossible because

$$E_{eff} = \varnothing.$$

Thus, no proper subset of $V(C_n)$ can be an effective dominating set.

Therefore, C_n admits no proper effective dominating set, while the full vertex set $V(C_n)$ is an effective dominating set.

Proposition 3.21 Let $G = K_{1,n}$ be a star $\mathcal{NUG}$ with center c and leaves

$$v_1, v_2, \ldots, v_n.$$

If every centre–leaf edge is effective, then every leaf is effectively dominated by c.
Proof. Let

$$v_i, \quad 1 \leq i \leq n,$$

be an arbitrary leaf of G. Since $G = K_{1,n}$ is a star with center c, the vertex v_i is adjacent only to the center c. Thus,

$$cv_i \in E(G).$$

By hypothesis, every centre–leaf edge is effective. In particular,

$$cv_i$$

is an effective edge.
Therefore, the leaf v_i has an effective neighbor c. Hence, v_i is effectively dominated by c.
Since v_i was arbitrary, every leaf

$$v_1, v_2, \ldots, v_n$$

is effectively dominated by the centre c.

Proposition 3.22 Let $G = K_{1,n}$ be a star $\mathcal{NUG}$ with center c. If $c \in F$ and at least one leaf remains in $G - F$, then $\{c\} - F = \varnothing$ cannot effectively dominate the surviving leaf under the open-neighborhood interpretation.
Proof. Let $G = K_{1,n}$ be a star $\mathcal{NUG}$ with center c, and let

$$F \subseteq V(G)$$

be a failure set such that

$$c \in F.$$

Consider the set

$$D = \{c\}.$$

Since $c \in F$, we have

$$D - F = \{c\} - F = \varnothing.$$

By hypothesis, at least one leaf, say v_i, remains in the resulting graph $G - F$. Thus,

$$v_i \in V(G - F) \backslash (D - F).$$

Under the open-neighbourhood interpretation of effective domination, a vertex outside the dominating set must have an effective neighbour belonging to the dominating set. However,

$$D - F = \varnothing,$$

so there is no vertex in $D - F$ that can effectively dominate v_i.
Therefore, the surviving leaf v_i is not effectively dominated by $D - F$. Hence,

$$\{c\} - F = \varnothing$$

cannot be an effective dominating set of $G - F$.

Theorem 3.12 Let $G = K_{1,n}$ be a star $\mathcal{NUG}$ with $n \geq 2$. If resilience is required with respect to arbitrary vertex failures, including failure of the center, then no set of the form

$$D = \{c\}$$

is a 1-resilient effective dominating set.
Proof. Let

$$D = \{c\}$$

and consider the failure set

$$F = \{c\}.$$

Since

$$|F| = 1,$$

this is an admissible failure set for 1-resilience.
After the failure of the centre vertex c, we have

$$D - F = \{c\} - \{c\} = \varnothing.$$

On the other hand, since $G = K_{1,n}$ with $n \geq 2$, all n leaves remain in $G - F$. In particular,

$$V(G - F) \neq \varnothing.$$

Let v_i be any surviving leaf. Then

$$v_i \in V(G - F) \backslash (D - F).$$

Under the open-neighbourhood interpretation of effective domination, v_i must have an effective neighbor in $D - F$ in order to be effectively dominated. However,

$$D - F = \varnothing,$$

so no such effective neighbour exists.
Therefore,

$$D - F$$

is not an effective dominating set of $G - F$. Hence, $D = \{c\}$ fails the defining condition for a 1-resilient effective dominating set.
Thus, no set of the form

$$D = \{c\}$$

is a 1-resilient effective dominating set when arbitrary vertex failures, including failure of the centre, are allowed.

Proposition 3.23 Let $G = K_{m,n}$ be a complete bipartite $\mathcal{NUG}$ with partite sets

$$X = \{x_1, x_2, \dots, x_m\}, \qquad Y = \{y_1, y_2, \dots, y_n\}.$$

If every edge joining X and Y is effective and the effective-edge property is preserved under every admissible vertex deletion, then $X - F$ effectively dominates $G - F$ for every

$$F \subseteq V, \qquad |F| \leq k,$$

provided that

$$m \geq k + 1.$$

Proof. Let

$$F \subseteq V, \qquad |F| \leq k$$

be an arbitrary admissible failure set.
Since

$$|X| = m \geq k + 1$$

and at most k vertices are deleted, we have

$$|X - F| \geq m - |F| \geq m - k \geq 1.$$

Therefore,

$$X - F \neq \varnothing.$$

Now let

$$y \in V(G - F) \backslash (X - F).$$

Since X and Y are the two partite sets of the complete bipartite graph, every vertex outside $X - F$ that survives in $G - F$ belongs to $Y - F$. Hence,

$$y \in Y - F.$$

Because

$$X - F \neq \varnothing,$$

choose any vertex

$$x \in X - F.$$

Since G is complete bipartite,

$$xy \in E(G).$$

By hypothesis, every edge joining X and Y is effective. Moreover, the effective-edge property is preserved under every admissible vertex deletion. Since

$$|F| \leq k,$$

the edge xy remains effective in $G - F$.
Thus, every vertex

$$y \in V(G - F) \backslash (X - F)$$

has an effective neighbour

$$x \in X - F.$$

Therefore,

$$X - F$$

is an effective dominating set of $G - F$.
Since F was arbitrary, $X - F$ effectively dominates $G - F$ for every

$$F \subseteq V, \qquad |F| \leq k.$$

Hence, the proposition follows.

Proposition 3.24 Let $G = K_{m,n}$ be a complete bipartite $\mathcal{NUG}$ satisfying the effective-edge preservation condition. If

$$m \geq k + 1,$$

then

$$\gamma_{RE}^{N,k}(G) \leq m.$$

Proof. Let

$$X = \{x_1, x_2, \dots, x_m\}$$

be the partite set of G having cardinality m.
Since

$$m \geq k + 1,$$

the preceding proposition implies that for every failure set

$$F \subseteq V(G), \qquad |F| \leq k,$$

the set

$$X - F$$

is an effective dominating set of $G - F$. Therefore, X is a k-resilient effective dominating set of G.

By the definition of $\gamma_{RE}^{N,k}(G)$ as the minimum cardinality of a k-resilient effective dominating set of G, we have

$$\gamma_{RE}^{N,k}(G) \leq |X| = m.$$

Hence,

$$\gamma_{RE}^{N,k}(G) \leq m.$$

Proposition 3.25 Let $G = K_{m,n}$ be a complete bipartite $\mathcal{NUG}$ satisfying the effective-edge preservation condition. If

$$n \geq k + 1,$$

then

$$\gamma_{RE}^{N,k}(G) \leq n.$$

Proof. Let

$$Y = \{y_1, y_2, \dots, y_n\}$$

be the partite set of G having cardinality n.

Since

$$n \geq k + 1,$$

for every failure set

$$F \subseteq V(G), \qquad |F| \leq k,$$

we have

$$|Y - F| \geq n - k \geq 1.$$

Thus, at least one vertex of Y survives every admissible failure set.

Let $x \in X - F$ be any vertex outside $Y - F$. Since $G = K_{m,n}$ is complete bipartite, x is adjacent to every vertex of Y, and hence to every vertex of $Y - F$. By the effective-edge preservation condition, each surviving edge

$$xy, \qquad y \in Y - F,$$

remains effective in $G - F$. Therefore, every vertex in $X - F$ is effectively dominated by a vertex in $Y - F$.

Hence, $Y - F$ is an effective dominating set of $G - F$ for every

$$F \subseteq V(G), \qquad |F| \leq k.$$

Therefore, Y is a k-resilient effective dominating set of G.

By the definition of $\gamma_{RE}^{N,k}(G)$,

$$\gamma_{RE}^{N,k}(G) \leq |Y| = n.$$

Thus,

$$\gamma_{RE}^{N,k}(G) \leq n.$$

Theorem 3.13 Let $G = K_{m,n}$ be a complete bipartite $\mathcal{NUG}$ satisfying the effective-edge preservation condition. If

$$m \geq k + 1, \qquad n \geq k + 1,$$

then

$$\gamma_{RE}^{N,k}(G) \leq \min\{m, n\}.$$

Proof. Let X and Y be the two partite sets of G, with

$$|X| = m, \qquad |Y| = n.$$

Since

$$m \geq k + 1,$$

the preceding proposition implies that X is a k-resilient effective dominating set of G. Hence,

$$\gamma_{RE}^{N,k}(G) \leq |X| = m.$$

Similarly, since

$$n \geq k + 1,$$

the other partite set Y is also a k-resilient effective dominating set of G. Therefore,

$$\gamma_{RE}^{N,k}(G) \leq |Y| = n.$$

Consequently, $\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G})$ is bounded above by both m and n. Thus,

$$\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G}) \leq \min\{\mathrm{m}, \mathrm{n}\}.$$

Hence, the result follows.

Theorem 3.14 Let $\mathrm{G} = \mathrm{K}_{\mathrm{m,n}}$ be a complete bipartite $\mathcal{NUG}$ satisfying the effective-edge preservation condition. If

$$\min\{\mathrm{m}, \mathrm{n}\} = \mathrm{k} + 1$$

and the smaller partite set is a k-resilient effective dominating set, then

$$\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G}) = \mathrm{k} + 1.$$

Proof. Let S denote the smaller partite set of G. Since

$$\min\{\mathrm{m}, \mathrm{n}\} = \mathrm{k} + 1,$$

we have

$$|\mathrm{S}| = \mathrm{k} + 1.$$

By assumption, S is a k-resilient effective dominating set of G. Hence, by the definition of the resilient $\mathcal{EDN}$,

$$\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G}) \leq |\mathrm{S}| = \mathrm{k} + 1.$$

On the other hand, since $|\mathrm{S}| = \mathrm{k} + 1$ and S is a k-resilient effective dominating set, the general lower bound for the resilient $\mathcal{EDN}$ gives

$$\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G}) \geq \mathrm{k} + 1.$$

Combining the above inequalities, we obtain

$$\mathrm{k} + 1 \leq \gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G}) \leq \mathrm{k} + 1.$$

Therefore,

$$\gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G}) = \mathrm{k} + 1.$$

Proposition 3.26 Let G be a $\mathcal{NUG}$. If D is a k-resilient effective dominating set, then D is an r-resilient effective dominating set for every integer

$$0 \leq \mathrm{r} \leq \mathrm{k}.$$

Proof. Since D is a k-resilient effective dominating set, by definition, for every failure set

$$\mathrm{F} \subseteq \mathrm{V}(\mathrm{G}), \qquad |\mathrm{F}| \leq \mathrm{k},$$

the set

$$\mathrm{D} - \mathrm{F}$$

is an effective dominating set of the remaining graph

$$\mathrm{G} - \mathrm{F}.$$

Let r be any integer satisfying

$$0 \leq \mathrm{r} \leq \mathrm{k}.$$

Consider an arbitrary failure set

$$F \subseteq V(G)$$

with

$$|F| \leq r.$$

Since $r \leq k$, we have

$$|F| \leq r \leq k.$$

Therefore, F is also an admissible failure set under the definition of k-resilience. Hence,

$$D - F$$

is an effective dominating set of

$$G - F.$$

Since the above holds for every failure set $F \subseteq V(G)$ satisfying $|F| \leq r$, it follows that D is an r-resilient effective dominating set of G.
Therefore,

$$\text{Disanr} - \text{resilienteffectivedominatingsetforevery} 0 \leq r \leq k.$$

Theorem 3.15 Let G be a $\mathcal{NUG}$. If G admits a k-resilient effective dominating set for some positive integer k, then

$$\gamma_{RE}^{N,r}(G) \leq \gamma_{RE}^{N,k}(G)$$

for every integer

$$0 \leq r \leq k.$$

Proof. Since G admits a k-resilient effective dominating set, the resilient $\mathcal{EDN}$

$$\gamma_{RE}^{N,k}(G)$$

is well-defined. Let D_k be a $\gamma_{RE}^{N,k}$-set of G. Thus,

$$|D_k| = \gamma_{RE}^{N,k}(G),$$

and D_k is a k-resilient effective dominating set.
Let r be any integer satisfying

$$0 \leq r \leq k.$$

By the preceding proposition, every k-resilient effective dominating set is also an r-resilient effective dominating set. Hence, D_k is an r-resilient effective dominating set of G.
Therefore, by the definition of the minimum resilient $\mathcal{EDN}$,

$$\gamma_{RE}^{N,r}(G) \leq |D_k|.$$

Since

$$|D_k| = \gamma_{RE}^{N,k}(G),$$

we obtain

$$\gamma_{RE}^{N,r}(G) \leq \gamma_{RE}^{N,k}(G).$$

Thus,

$$\gamma_{\mathrm{RE}}^{\mathrm{N,r}}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{\mathrm{N,k}}(\mathrm{G})$$

for every integer r satisfying $0 \leq \mathrm{r} \leq \mathrm{k}$.

Proposition 3.27 Let G be a $\mathcal{NUG}$ and let D be a k-resilient effective dominating set. For every vertex

$$\mathrm{v} \in \mathrm{V}\backslash\mathrm{D},$$

the vertex v has an effective neighbor in $\mathrm{D} - \mathrm{F}$ for every failure set

$$\mathrm{F} \subseteq \mathrm{V}, \qquad |\mathrm{F}| \leq \mathrm{k}, \qquad \mathrm{v} \notin \mathrm{F}.$$

Proof. Let

$$\mathrm{v} \in \mathrm{V}\backslash\mathrm{D}$$

be arbitrary, and let

$$\mathrm{F} \subseteq \mathrm{V}, \qquad |\mathrm{F}| \leq \mathrm{k}, \qquad \mathrm{v} \notin \mathrm{F}$$

be any admissible failure set.

Since D is a k-resilient effective dominating set, by definition,

$$\mathrm{D} - \mathrm{F}$$

is an effective dominating set of the remaining graph

$$\mathrm{G} - \mathrm{F}.$$

Since $\mathrm{v} \notin \mathrm{F}$ and $\mathrm{v} \notin \mathrm{D}$, we have

$$\mathrm{v} \in \mathrm{V}(\mathrm{G} - \mathrm{F})\backslash(\mathrm{D} - \mathrm{F}).$$

Therefore, by the definition of effective domination, there exists a vertex

$$\mathrm{u} \in \mathrm{D} - \mathrm{F}$$

such that u is an effective neighbor of v in $\mathrm{G} - \mathrm{F}$. Equivalently,

$$\mathrm{uv} \in \mathrm{E}(\mathrm{G} - \mathrm{F})$$

and

$$\mathrm{uv}$$

is an effective edge of $\mathrm{G} - \mathrm{F}$.

Thus, every vertex $\mathrm{v} \in \mathrm{V}\backslash\mathrm{D}$ has an effective neighbor in $\mathrm{D} - \mathrm{F}$ for every failure set $\mathrm{F} \subseteq \mathrm{V}$ satisfying

$$|\mathrm{F}| \leq \mathrm{k}, \qquad \mathrm{v} \notin \mathrm{F}.$$

Hence, the result follows.

Theorem 3.16 Let G be a $\mathcal{NUG}$ and let D be an effective dominating set. Suppose that, under every vertex deletion, no new effective neighbor of a surviving vertex is created. If a vertex

$$\mathrm{v} \in \mathrm{V}\backslash\mathrm{D}$$

has at most k effective neighbors in D, then D is not a k-resilient effective dominating set.

Proof. Let

$$\mathrm{N_E}(\mathrm{v}) \cap \mathrm{D} = \{\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_\mathrm{t}\},$$

where $\mathrm{N_E}(\mathrm{v})$ denotes the set of effective neighbors of v, and suppose that

$$t \leq k.$$

Define the failure set

$$F = N_E(v) \cap D.$$

Then

$$F \subseteq D, \qquad |F| = t \leq k.$$

Since $v \notin D$ and $F \subseteq D$, we have

$$v \notin F.$$

Hence, v survives in the remaining graph $G - F$.

Moreover, by the construction of F, every effective neighbor of v that belongs to D has been removed. Thus,

$$N_E(v) \cap (D - F) = \varnothing.$$

By the assumed condition that vertex deletion does not create any new effective neighbor of a surviving vertex, v has no effective neighbor in $D - F$ in $G - F$.
Therefore, v is not effectively dominated by $D - F$ in $G - F$. Consequently,

$$D - F$$

is not an effective dominating set of $G - F$.
Since

$$F \subseteq V, \qquad |F| \leq k,$$

this contradicts the requirement for k-resilience. Hence,

$$Disnotak - resilienteffectivedominatingset.$$

Corollary 3.1 Let G be a $\mathcal{NUG}$ and let D be a k-resilient effective dominating set. Suppose that, under every vertex deletion, no new effective neighbour of a surviving vertex is created. Then every vertex

$$v \in V \backslash D$$

has at least $k + 1$ distinct effective neighbors in D.
Proof. Let

$$v \in V \backslash D.$$

Suppose, to the contrary, that v has at most k distinct effective neighbors in D. Then

$$|N_E(v) \cap D| \leq k.$$

By the preceding theorem, under the assumed effective-neighbour preservation condition, D cannot be a k-resilient effective dominating set.
This contradicts the hypothesis that D is a k-resilient effective dominating set. Therefore,

$$|N_E(v) \cap D| \geq k + 1.$$

Hence, every vertex

$$v \in V \backslash D$$

has at least $k + 1$ distinct effective neighbors in D.

Theorem 3.17 Let G be a $\mathcal{NUG}$ and let D be an effective dominating set. Suppose that every vertex in

$$V\backslash D$$

has at least $k+1$ distinct effective neighbours in D, and the effective-edge property is preserved under every deletion of at most k vertices. Then D is a k-resilient effective dominating set.
Proof. Let

$$F \subseteq V$$

be any failure set satisfying

$$|F| \leq k.$$

We show that

$$D - F$$

is an effective dominating set of

$$G - F.$$

Let

$$v \in V(G - F)\backslash(D - F).$$

Since $v \notin F$, we have

$$v \in V\backslash D.$$

By hypothesis, v has at least $k+1$ distinct effective neighbors in D. Let these effective neighbors be

$$u_1, u_2, \dots, u_{k+1} \in D.$$

Since

$$|F| \leq k,$$

at most k of the vertices

$$u_1, u_2, \dots, u_{k+1}$$

can belong to F. Therefore, at least one of these effective neighbors, say u_j, survives the failure set. Hence,

$$u_j \in D - F.$$

Moreover, since the effective-edge property is preserved under every deletion of at most k vertices, the edge

$$u_j v$$

remains effective in $G - F$. Thus, v has an effective neighbor in $D - F$.

Since $v \in V(G - F)\backslash(D - F)$ was arbitrary, every vertex outside $D - F$ is effectively dominated by $D - F$. Consequently,

$$D - F$$

is an effective dominating set of $G - F$.
As this holds for every failure set

$$F \subseteq V, \qquad |F| \leq k,$$

it follows that D is a k-resilient effective dominating set of G.
Therefore,

$$\text{Disak} - \text{resilienteffectivedominatingset.}$$

4. NUMERICAL APPLICATION TO RESILIENT HEALTHCARE RESOURCE ALLOCATION

Healthcare resource allocation requires the identification of strategically important healthcare units that can continue to provide operational support when one or more units become unavailable. In an emergency healthcare environment, the failure of a single unit may occur because of equipment malfunction, temporary staff shortage, excessive patient load, maintenance activities, or communication disruption. Therefore, selecting healthcare units only according to their normal operational importance may not provide sufficient protection against unexpected failures. A resilient resource-allocation strategy should identify a minimum set of healthcare units such that the remaining units continue to receive effective support after the failure of any admissible healthcare unit.

To illustrate the practical usefulness of the proposed resilient effective domination framework, consider a healthcare network consisting of five interconnected units,

$$V = \{H_1, H_2, H_3, H_4, H_5\},$$

where

$$\begin{aligned} H_1 &: \text{EmergencyDepartment}, \\ H_2 &: \text{IntensiveCareUnit}, \\ H_3 &: \text{CardiologyUnit}, \\ H_4 &: \text{DiagnosticLaboratory}, \\ H_5 &: \text{PharmacyUnit}. \end{aligned}$$

An edge between two healthcare units represents a direct resource-sharing or operational-support relationship. Such a relationship may include emergency referrals, sharing of medical equipment, diagnostic information exchange, medication support, or transfer of patients during periods of excessive demand. The uncertain operational condition of each healthcare relationship is represented by a neutrosophic triplet

$$\mathcal{N}(H_iH_j) = (T_{ij}, I_{ij}, F_{ij}),$$

where T_{ij} represents the degree of reliable interaction, I_{ij} represents the degree of indeterminacy, and F_{ij} represents the degree of unreliable interaction.

The neutrosophic triplet values assigned to the healthcare units and their interconnections are hypothetical and are assumed for numerical illustration. They do not represent actual healthcare records or values obtained through expert judgment. The triplets are constructed to demonstrate how uncertainty in healthcare resource relationships can be represented using neutrosophic values and how the proposed resilient effective domination framework can be applied to identify a minimum set of healthcare units that maintains effective resource support under vertex failures. Accordingly, the numerical results illustrate the mathematical applicability of the proposed framework rather than an empirical evaluation of an actual healthcare network. The numerical data considered for the healthcare network are summarised in Table 1.

Table 1
Assumed Neutrosophic Values for the Healthcare Relationships

Healthcare Relationship	T_{ij}	I_{ij}	F_{ij}
H_1H_2	0.94	0.04	0.02
H_2H_3	0.91	0.06	0.03
H_3H_4	0.89	0.07	0.04
H_4H_5	0.93	0.05	0.02

Using the strength function defined for the $\mathcal{NUG}$,

$$\mathrm{str}(H_iH_j) = \frac{T_{ij} + (1 - I_{ij}) + (1 - F_{ij})}{3},$$

the strength of the relationship H_1H_2 is

$$\begin{aligned} \mathrm{str}(H_1H_2) &= \frac{0.94 + (1 - 0.04) + (1 - 0.02)}{3} \\ &= \frac{0.94 + 0.96 + 0.98}{3} \\ &= 0.9600. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{str}(H_2H_3) &= \frac{0.91 + (1 - 0.06) + (1 - 0.03)}{3} \\ &= \frac{0.91 + 0.94 + 0.97}{3} \\ &= 0.9400, \\ \mathrm{str}(H_3H_4) &= \frac{0.89 + (1 - 0.07) + (1 - 0.04)}{3} \\ &= \frac{0.89 + 0.93 + 0.96}{3} \\ &= 0.9267, \end{aligned}$$

and

$$\begin{aligned} \mathrm{str}(H_4H_5) &= \frac{0.93 + (1 - 0.05) + (1 - 0.02)}{3} \\ &= \frac{0.93 + 0.95 + 0.98}{3} \\ &= 0.9533. \end{aligned}$$

The selected edge-strength function employs an arithmetic aggregation of the truth, indeterminacy and falsity components characterising a neutrosophic relationship. Specifically, the truth-membership component Tij contributes directly to the resulting strength, whereas the indeterminacy and falsity components are incorporated through their respective complementary values, (1-Iij) and (1-Fij). Accordingly, an increase in truth membership, accompanied by reductions in indeterminacy and falsity, leads to a higher edge-strength value. The arithmetic mean is used as a straightforward and consistent aggregation approach for converting these three components into a single scalar measure. This enables quantitative comparisons among healthcare relationships and facilitates the identification of effective edges in accordance with the specified effectiveness criterion Therefore, the resulting numerical strength values are as follows.

Table 2
Calculated Strengths of the Healthcare Relationships

Healthcarerelationship	$\mathrm{str}(H_iH_j)$
H_1H_2	0.9600
H_2H_3	0.9400
H_3H_4	0.9267
H_4H_5	0.9533

4.1 Sensitivity Analysis

To assess the sensitivity of the proposed healthcare resource allocation model to changes in the neutrosophic triplet components, two alternative perturbation scenarios are investigated. In the first scenario, the truth-membership component of each healthcare relationship is raised by 0.01, whereas both its indeterminacy and falsity components are reduced by 0.005. Conversely, in the second scenario, the truth-membership value is lowered by 0.01, while the corresponding indeterminacy and falsity values are each increased by 0.005. Both perturbation schemes preserve the aggregate sum of the three components and ensure that their values remain within the admissible ranges.

For each scenario, the corresponding edge strengths are recalculated using the same strength function established in this study. The original and perturbed strength values are reported in Table. The findings indicate that the numerical strengths of the edges vary in response to the imposed perturbations. Nevertheless, the effective-edge classification of individual healthcare relationships remains unaffected.

This invariance arises from the path structure of the healthcare network, in which no edge has an alternative path connecting its endpoints. Consequently, the effective-edge configuration is preserved, and the minimum single-failure resilient effective dominating set remains D ={H1,H3,H5} , with cardinality three, across both perturbation scenarios.

Table 3
Sensitivity Analysis of Healthcare Relationship Strengths under Perturbed Neutrosophic Triplet Values

Healthcare Relationship	Original Strength	Increased Truth Scenario	Decreased Truth Scenario
H_1H_2	0.9600	0.9667	0.9533
H_2H_3	0.9400	0.9467	0.9333
H_3H_4	0.9267	0.9333	0.9200
H_4H_5	0.9533	0.9600	0.9467

In all three scenarios, the minimum single-failure resilient effective domination number remains ($\gamma_E^N(G)$=3), demonstrating the stability of the selected healthcare units under the considered variations in the neutrosophic triplet values.

The resulting healthcare network is

$$G = (V, E),$$

where

$$E = \{H_1H_2, H_2H_3, H_3H_4, H_4H_5\}.$$

Thus, the healthcare network has the structure

$$H_1 - H_2 - H_3 - H_4 - H_5.$$

Since G is a path, there is no alternative path between the two endpoints of any edge.

Consequently, every edge in E is an effective edge. Hence,

$$\mathrm{E_E} = \{\mathrm{H_1H_2}, \mathrm{H_2H_3}, \mathrm{H_3H_4}, \mathrm{H_4H_5}\}.$$

The effective-neighbour sets are therefore

$$\begin{aligned} \mathrm{N_E}(\mathrm{H_1}) &= \{\mathrm{H_2}\}, \\ \mathrm{N_E}(\mathrm{H_2}) &= \{\mathrm{H_1}, \mathrm{H_3}\}, \\ \mathrm{N_E}(\mathrm{H_3}) &= \{\mathrm{H_2}, \mathrm{H_4}\}, \\ \mathrm{N_E}(\mathrm{H_4}) &= \{\mathrm{H_3}, \mathrm{H_5}\}, \\ \mathrm{N_E}(\mathrm{H_5}) &= \{\mathrm{H_4}\}. \end{aligned}$$

We first determine the ordinary $\mathcal{EDN}$. Consider

$$\mathrm{D_0} = \{\mathrm{H_2}, \mathrm{H_4}\}.$$

For the vertices outside $\mathrm{D_0}$, we have

$$\mathrm{H_1} \sim_{\mathrm{E}} \mathrm{H_2}, \qquad \mathrm{H_3} \sim_{\mathrm{E}} \mathrm{H_2}, \qquad \mathrm{H_5} \sim_{\mathrm{E}} \mathrm{H_4}.$$

Therefore, $\mathrm{D_0}$ is an effective dominating set of G, and

$$\gamma_{\mathrm{E}}^{\mathrm{N}}(\mathrm{G}) \leq 2.$$

No single vertex can effectively dominate all the remaining healthcare units. In particular,

$$|\mathrm{N_E}(\mathrm{H_2})| = 2, \qquad |\mathrm{N_E}(\mathrm{H_3})| = 2, \qquad |\mathrm{N_E}(\mathrm{H_4})| = 2,$$

while

$$|\mathrm{N_E}(\mathrm{H_1})| = |\mathrm{N_E}(\mathrm{H_5})| = 1.$$

Hence, a single healthcare unit cannot provide effective coverage for all other units. Therefore,

$$\gamma_{\mathrm{E}}^{\mathrm{N}}(\mathrm{G}) = 2.$$

Although two healthcare units are sufficient under normal operating conditions, this selection does not provide resilience against the failure of one healthcare unit. To demonstrate this, let

$$\mathrm{D_0} = \{\mathrm{H_2}, \mathrm{H_4}\}.$$

Consider the failure of the Intensive Care Unit,

$$\mathrm{F} = \{\mathrm{H_2}\}.$$

The remaining selected healthcare unit is

$$\mathrm{D_0} - \mathrm{F} = \{\mathrm{H_4}\}.$$

The Emergency Department $\mathrm{H_1}$ remains in the healthcare network, but its only effective neighbor is $\mathrm{H_2}$. Since

$$\mathrm{H_2} \in \mathrm{F},$$

the surviving Emergency Department has no effective neighbour in the surviving selected set. Therefore, $\mathrm{D_0} - \mathrm{F}$ is not an effective dominating set of the surviving healthcare network. Hence,

$$\mathrm{D_0}$$

is not a 1-resilient effective dominating set.
We now determine a resilient resource-allocation set. Consider

$$\mathrm{D_1} = \{\mathrm{H_2}, \mathrm{H_3}, \mathrm{H_4}\}.$$

For the vertices outside D_1,

$$V \backslash D_1 = \{H_1, H_5\}.$$

The Emergency Department H_1 has the effective neighbor

$$H_2 \in D_1,$$

while the Pharmacy Unit H_5 has the effective neighbor

$$H_4 \in D_1.$$

Hence, D_1 is an effective dominating set.
However, because the required resilience level is

$$k = 1,$$

we must examine the failure of each individual healthcare unit.
First, suppose

$$F_1 = \{H_2\}.$$

The surviving selected healthcare units are

$$D_1 - F_1 = \{H_3, H_4\}.$$

The surviving vertices outside this set are

$$\{H_1, H_5\}.$$

The Pharmacy Unit H_5 remains effectively supported by H_4, but the Emergency Department H_1 has no surviving effective neighbor because its only effective neighbor H_2 has failed. Thus,

$$D_1$$

is not 1-resilient.
Therefore, the set $\{H_2, H_3, H_4\}$ is also insufficient for the required resilience level.
We consequently examine a set containing both end-support units,

$$D_2 = \{H_1, H_2, H_4, H_5\}.$$

Here,

$$|D_2| = 4,$$

and the only healthcare unit outside D_2 is

$$H_3.$$

The vertex H_3 has two distinct effective neighbours,

$$H_2, \quad H_4,$$

both belonging to D_2. Thus,

$$|N_E(H_3) \cap D_2| = 2 = k + 1.$$

This numerical redundancy is precisely what is required for tolerance against one selected-unit failure. If H_2 fails, then

$$H_3$$

remains effectively connected to H_4. If H_4 fails, then

$$H_3$$

remains effectively connected to H_2.
The failure cases involving the other selected healthcare units can also be evaluated directly.
For

$$F_1 = \{H_1\},$$

the surviving selected set is

$$D_2 - F_1 = \{H_2, H_4, H_5\},$$

and the only surviving vertex outside it is H_3. Since

$$H_3H_2 \in E_E,$$

the vertex H_3 remains effectively dominated.
For

$$F_2 = \{H_2\},$$

we have

$$D_2 - F_2 = \{H_1, H_4, H_5\}.$$

Again, the only surviving vertex outside the selected set is H_3, and

$$H_3H_4 \in E_E.$$

Hence, H_3 remains effectively dominated.
For

$$F_3 = \{H_4\},$$

we have

$$D_2 - F_3 = \{H_1, H_2, H_5\},$$

and

$$H_3H_2 \in E_E.$$

Thus, H_3 remains effectively dominated.
Finally, for

$$F_4 = \{H_5\},$$

we have

$$D_2 - F_4 = \{H_1, H_2, H_4\},$$

and both

$$H_3H_2 \in E_E$$

and

$$H_3H_4 \in E_E$$

remain available.

Since every possible single-vertex failure leaves H_3 effectively dominated by at least one surviving selected healthcare unit, D_2 is a 1-resilient effective dominating set.

The failure analysis is summarized in Table 4.

Table 4
Single-Vertex Failure Analysis of the Resilient Healthcare Support Set

Failureset	$D_2 - F$	Surviving Outside Vertex	Effective Support
$\{H_1\}$	$\{H_2, H_4, H_5\}$	H_3	H_3H_2
$\{H_2\}$	$\{H_1, H_4, H_5\}$	H_3	H_3H_4
$\{H_4\}$	$\{H_1, H_2, H_5\}$	H_3	H_3H_2
$\{H_5\}$	$\{H_1, H_2, H_4\}$	H_3	H_3H_2

4.2 Verification of the Minimum Resilient Effective Dominating Set

The next step is to establish the minimum cardinality of a 1-resilient effective dominating set for the healthcare network. Although the minimum effective dominating set under normal operating conditions consists of two vertices, this configuration fails to meet the prescribed resilience requirement, as evidenced by the preceding analysis of vertex failures. Within the framework of single-vertex failure tolerance, consider the following three-element set

$$D = \{H_1, H_3, H_5\}.$$

The vertices outside (D) are (H_2) and (H_4). Since the healthcare network has the path structure (H_1-H_2-H_3-H_4-H_5), the vertex (H_2) is adjacent to (H_1) and (H_3), whereas (H_4) is adjacent to (H_3) and (H_5). Thus, each vertex outside (D) has two effective neighbours in (D).

Consider the failure of each vertex in the network. If (H_1) fails, (H_2) remains effectively dominated by (H_3), while (H_4) remains effectively dominated by (H_3) and (H_5). If (H_2) fails, the remaining outside vertex (H_4) retains its effective neighbours (H_3) and (H_5). If (H_3) fails, (H_2) remains effectively dominated by (H_1), and (H_4) remains effectively dominated by (H_5). If (H_4) fails, (H_2) retains its effective neighbors (H_1) and (H_3). Finally, if (H_5) fails, (H_2) remains effectively dominated by (H_1) and (H_3), while (H_4) remains effectively dominated by (H_3). Therefore, (D) satisfies Definition 3.9 and is a (1)-resilient effective dominating set.

The minimality of this set must now be demonstrated. A singleton set cannot satisfy the 1-resilient effective domination requirement because the removal of its only selected vertex leaves no remaining selected vertex capable of dominating the network. Assume, for contradiction, that a 1-resilient effective dominating set consists of precisely two vertices. If one selected vertex, say (u), fails, the resulting dominating set contains only the other vertex, (v). According to Definition 3.9, every vertex in the network must then be connected to (v) through an effective edge. Given that the network comprises five vertices, this condition necessitates that (v) has a minimum of three neighbours. Nevertheless, the maximum degree of the path (P_5) is two, contradicting the required condition. Therefore, no set with fewer than three vertices can fulfil the single-vertex failure resilience requirement.

Consequently, the minimum cardinality of a (1)-resilient effective dominating set for the healthcare network is three. Thus,

$$\gamma_{RE}^{N,1}(G) = 3.$$

The numerical comparison between ordinary and resilient healthcare resource allocation is therefore

$$\gamma_{E}^{N}(G) = 2 < \gamma_{RE}^{N,1}(G) = 3.$$

This difference has a direct healthcare-management interpretation. Under normal operating conditions, only two healthcare units are sufficient to provide effective resource coverage:

$$D_0 = \{H_2, H_4\}.$$

However, this allocation is vulnerable to the failure of either selected support unit. To tolerate the failure of any one healthcare unit, the resource-allocation strategy must be expanded to

$$\mathrm{D} = \{\mathrm{H}_1, \mathrm{H}_3, \mathrm{H}_5\}.$$

Accordingly, the designated healthcare facilities comprise the Emergency Department, Cardiology Unit, and Pharmacy Unit. Under this arrangement, the Intensive Care Unit is effectively supported by both the Emergency Department and the Cardiology Unit, whereas the Diagnostic Laboratory receives effective support from the Cardiology Unit and Pharmacy Unit. Therefore, each healthcare unit excluded from the selected configuration has two effective supporting units. This redundancy ensures that the failure of any single selected healthcare unit does not entirely disrupt the effective resource-support system, as at least one effective supporting unit remains available to sustain each surviving non-selected unit. The numerical results demonstrate that ordinary effective domination and resilient effective domination lead to different management decisions:

Decisioncriterion	Selectedhealthcareunits	Numberofunits
Ordinaryeffectivedomination	$\{\mathrm{H}_2, \mathrm{H}_4\}$	2
$1-$ resilienteffectivedomination	$\{\mathrm{H}_1, \mathrm{H}_3, \mathrm{H}_5\}$	3

The additional one selected unit represents the numerical cost of resilience.

$$\frac{3-2}{2} \times 100 = 50\%$$

The transition from two to three selected healthcare units represents a 50% increase in the number of strategically allocated units. This supplementary allocation introduces resilience against the failure of any single healthcare unit whilst preserving effective resource support for all remaining non-selected units.

Accordingly, the proposed resilient effective domination framework establishes a quantitative basis for enabling healthcare administrators to reconcile resource utilisation efficiency with operational reliability. Rather than determining the selection of healthcare units exclusively on the basis of their performance under normal operating conditions, this approach identifies an expanded yet more robust resource-support configuration that sustains effective healthcare coverage despite unforeseen failures of individual units.

5. CONCLUSION

This study introduces a robust effective domination framework for neutrosophic under graphs by extending the classical notion of effective domination to incorporate resilience against vertex failures. The proposed k-resilient effective domination model establishes a foundation for investigating This monotonicity relations, domination bounds, vertex-deletion properties, neighbourhood-based characterisations, and structural theorems within several important classes of neutrosophic under graphs. In particular ,$\boldsymbol{\gamma_{\mathrm{RE}}^{\mathrm{N},0}(\mathrm{G}) \leq \gamma_{\mathrm{RE}}^{\mathrm{N},1}(\mathrm{G}) \leq \cdots \leq \gamma_{\mathrm{RE}}^{\mathrm{N},k}(\mathrm{G})}$, establish that increasing the degree of failure resilience necessitates a dominating configuration whose cardinality is no smaller than that required at lower levels of resilience. Furthermore, the analysis highlights the fundamental contribution of effective-neighbour redundancy to preserving domination when vertices are removed. The application to healthcare resource allocation illustrates the practical applicability of the proposed framework. Within the neutrosophic healthcare network under consideration, the conventional effective domination number is $\boldsymbol{\gamma_{\mathrm{E}}^{\mathrm{N}}(\mathrm{G}) = 2}$, whereas ensuring resilience against the failure of a single vertex requires $\boldsymbol{\gamma_{\mathrm{RE}}^{\mathrm{N},1}(\mathrm{G}) = 4}$. Thus, $\boldsymbol{\gamma_{\mathrm{E}}^{\mathrm{N}}(\mathrm{G}) <}$

$\boldsymbol{\gamma_{RE}^{N,1}(G)}$ demonstrates that a resource allocation strategy designed to achieve optimal performance under normal operating conditions may prove inadequate when failures occur. Accordingly, the proposed framework provides a quantitative approach to reconciling resource utilisation efficiency with operational resilience in healthcare networks characterised by uncertainty. Future investigations could extend the proposed model to encompass weighted and cost-constrained resilient effective domination, simultaneous and temporally varying failures, dynamic neutrosophic information, and the incorporation of artificial intelligence techniques alongside multi-criteria decision-making approaches. Such developments may facilitate more adaptive and robust resilience-planning strategies across healthcare, transportation, communication, supply-chain systems, and other critical infrastructure networks.

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