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Enhancing Network Security Solutions Techniques with Redefined Normal Bipolar Complex Fuzzy Soft Set Using a Multi-Criteria Group Decision-Making Methodology

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ABSTRACT

Effective network security requires the integration of carefully designed protective mechanisms with continuous monitoring to safeguard computer networks against evolving cyber threats. Five essential security strategies are considered to strengthen network resilience against conventional and emerging vulnerabilities, providing a systematic foundation for addressing contemporary cybersecurity challenges. Within this context, the present research introduces novel theoretical constructs and methodological perspectives intended to improve the representation and interpretation of complex, uncertain, and ambiguous information encountered in practical decision-making environments. Specifically, the study develops a reformulated framework of normal bipolar complex fuzzy soft sets by integrating the mathematical foundations of bipolar complex fuzzy sets and soft sets into a unified structure. The principal objective is to establish the operational foundations of aggregation mechanisms and examine the proposed framework through weighted averaging and geometric aggregation operators, together with their associated mathematical properties. Building upon these formulations, an innovative multi-criteria group decision-making approach is developed to evaluate alternative solutions for mitigating network security threats using the proposed aggregation operators. The mathematical consistency of these operators is investigated through fundamental algebraic properties, including boundedness, commutativity, idempotency, and monotonicity. Furthermore, a structured decision-making algorithm is formulated to enhance computational accessibility, practical applicability, and procedural simplicity relative to existing approaches. The effectiveness, robustness, and reliability of the proposed theoretical framework are subsequently assessed through sensitivity analysis, validity testing, and comparative evaluation. To demonstrate its operational relevance, the study incorporates a systematic decision-making procedure supported by practical examples illustrating the

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applicability and utility of the proposed sets in real-world decision environments. Finally, a comparative examination of the newly developed formulations and established theoretical approaches is undertaken to clarify their distinguishing characteristics and demonstrate the contributions of the proposed framework.

1. Introduction

Multi-criteria group decision-making (MCGDM) constitutes a systematic decision-support process through which a panel of knowledgeable experts assesses competing criteria to identify the most appropriate alternative from a range of available options. The application of multi-criteria decision-making (MCDM) methodologies for selecting optimal alternatives has been investigated in [27]. Within this broader analytical context, bipolar fuzzy set (BFS) theory extends the conventional fuzzy set (FS) framework by assigning two distinct membership assessments to each element, representing positive membership associated with support and negative membership reflecting opposition. This dual-membership representation facilitates the simultaneous consideration of conflicting evaluations and ambiguous information, particularly in complex decision-making (DM) environments. The theoretical framework can be extended to intuitionistic, complex, and soft-set environments while retaining a comprehensive algebraic foundation encompassing union, intersection, complementation, and aggregation operations (AO).

Moreover, BFS theory provides an important mathematical basis for established MCDM methodologies, including VIKOR and TOPSIS, enabling more refined alternative rankings through the combined interpretation of positive and negative membership assessments. Contemporary investigations have increasingly examined hybrid formulations integrating BFS theory with soft-set structures to accommodate multidimensional uncertainties and phase-dependent variations. Such developments have considerable practical significance across engineering, management science, and environmental planning, with applications encompassing supplier evaluation, energy resource management, healthcare logistics, and clustering problems. By incorporating favourable and unfavourable attributes within a unified mathematical representation, BFS theory offers a versatile foundation for constructing decision-support systems capable of generating more adaptable, interpretable, and reliable outcomes under real-world uncertainty and ambiguity.

1.1 Review of Literature

Mathematical modelling provides a structured mechanism for translating practical problems into formal mathematical representations, thereby supporting analytical evaluations and decisions grounded in explicitly defined information. Nevertheless, conventional modelling frameworks frequently encounter difficulties when representing imprecision, incomplete knowledge, and uncertainty. The foundational development of FS theory [2] established a mathematical approach for addressing such limitations and subsequently gained widespread applicability across numerous scientific disciplines. Further theoretical refinements extended the conventional FS formulation [29], while the introduction of intuitionistic fuzzy sets (IFS) [7] provided a more expressive representation of uncertainty by incorporating both membership grade (MG) and non-membership grade (NMG), subject to the condition that their combined value does not exceed image.

Subsequently, Pythagorean fuzzy sets (PFS) introduced an alternative formulation involving MG and NMG, with their sum constrained by a maximum value of image [34]. Building upon these developments, a DM framework incorporating complex PFS Hamacher AOs was formulated through the integration of PFS and complex FS theory [2]. Additional contributions established several aggregation mechanisms within the PFS environment [8], further expanding its mathematical and

computational capabilities. Moreover, various methodological approaches were introduced to address MCGDM problems under PFS-based uncertainty representations [6]. The authors in [24] changed the upper requirement from $0 \leq MG + NMG \leq 1$ to $0 \leq (MG)^2 + (NMG)^2 \leq 1$ to generate interval-valued PFSs (IVPFS). Th researchers [35] invented the q-rung orthopair FS, which overcomes the problems of both IFS and PFS. This notion is defined by two variables, MG and NMG, whose total powers must be within the range $[0,1]$. Wen-Ran [33] proposed the bipolar FS (BFS), which has a negative satisfaction rating in $[-1,0]$ and a positive satisfaction grade in $[0,1]$.

Although FS theory has provided a valuable mathematical foundation for representing uncertainty across diverse disciplines, the increasing complexity, ambiguity, and imprecision of practical decision environments have encouraged the development of more sophisticated fuzzy frameworks. Recent methodological advances include the application of sine trigonometric Fermatean normal fuzzy AOs to multiple-attribute decision-making (MADM) problems [22]. The theoretical foundations of bipolar fuzzy aggregation were subsequently established through the introduction of two fundamental operators: the bipolar fuzzy weighted geometric (BFWG) operator and the bipolar fuzzy weighted averaging (BFWA) operator [13]. Further extensions incorporated hesitant bipolar fuzzy weighted averaging and weighted geometric operators, thereby broadening the representation and aggregation of uncertain evaluations [31]. Additional investigations examined the mathematical characteristics of these aggregation mechanisms alongside the development of Hamacher-based AOs [32].

Beyond bipolar formulations, complex fuzzy sets (CFS) have emerged as an effective mathematical framework for representing ambiguous and uncertain information characterised by two-dimensional attributes. Within this framework, the Cartesian representation of satisfaction grades was investigated to enhance the interpretation of complex-valued membership information [28]. The underlying mathematical formulation extends MG from the conventional real-valued domain to the complex-number domain [26], allowing membership representations to incorporate information beyond the traditional one-dimensional structure. Consequently, whereas conventional FSs employ real-valued membership grades within a closed unit interval, CFSs accommodate complex-valued membership information associated with the unit disc in the complex plane. This generalisation was formalised through the extension of conventional FS theory to complex-valued membership structures [25], while subsequent investigations provided further theoretical clarification and interpretation of CFS formulations [37].

The integration of non-membership information subsequently led to the development of complex intuitionistic fuzzy sets (CIFS), in which both MG and NMG are represented within the unit circle of the complex plane [4]. Further advances in complex-valued uncertainty modelling included the introduction of complex root fuzzy sets (CROFS), extending the available mathematical approaches for representing and processing uncertain information [15]. The unit disc contains the picture value of $\varpi \in U$, and $\phi_C(\varpi)$ is the CFS C membership mapping. Thus, all values of $\phi_C(\varpi)$ exist inside a complex plane circle of radius one and are defined as $\phi_C(\varpi) = a_C(\varpi)e^{ip_C(\varpi)}$, where $i = \sqrt{-1}$. Phase and amplitude terms are the real-valued values $p_C(\varpi)$ and $a_C(\varpi)$ with $\phi_C(\varpi) \in [0,1]$. The symbol for the CFS C is $(\varpi, \phi_C(\varpi))|\varpi \in U$. The defining characteristic of CFSs is the incorporation of a phase component, which enables the representation of information beyond conventional real-valued membership structures. When this phase component is absent, the CFS formulation reduces to the standard FS framework. Existing literature demonstrates that CFS-based methodologies provide a more expressive alternative to conventional FS approaches, particularly when addressing complex problems involving two-dimensional information and uncertainty.

Soft-set theory, originally introduced to facilitate parameter-dependent evaluations of

alternatives [19], represents an extension of classical set-theoretic approaches and provides a flexible mathematical foundation for modelling uncertain information. Subsequent developments expanded its analytical capabilities through the introduction of advanced weighted geometric (WG) and weighted averaging (WA) operators [30]. Further theoretical progress integrated q-rung formulations with soft-set structures, establishing the concept of image-rung orthopair fuzzy soft sets (FSSs) [17]. Related investigations developed q-rung soft sets and examined their associated algebraic structures [11]. To strengthen the representation of uncertainty involving opposing membership assessments, bipolar intuitionistic fuzzy sets (BIFSs) were extended by incorporating bipolar MG and NMG information [9]. Similarly, bipolar Pythagorean fuzzy sets (BPFSs) impose a Pythagorean constraint on MG and NMG, thereby offering greater flexibility for MCDM problems involving conflicting evaluations. Additional generalisations introduced bipolar Fermatean fuzzy sets (BFFSs) [10].

In medical decision-support applications, a Type-II generalised IVPNS framework based on IVFSSs was investigated [21], while an innovative potential Pythagorean IVFSSs methodology was subsequently developed for MADM problems [16]. Further research extended similarity measures through bipolar complex fuzzy sets (BCFSs), demonstrating their relevance to pattern recognition, decision analysis, and system optimisation [18]. Dynamic BFS formulations incorporating AOs were also developed to support the evaluation and selection of emerging technologies within corporate integration environments [3]. Moreover, Dombi-based aggregation mechanisms were investigated under complex bipolar fuzzy sets (CBFSs) for DM applications [36], while aggregation procedures involving complex fuzzy Z-number sets were examined within MCDM frameworks [13]. The extension of these theoretical developments to multi-fuzzy bipolar soft sets (BFSS) provided additional mechanisms for representing information in DM problems [12]. Other significant generalisations include hyper fuzzy BFSSs [5] and N-BFSSs [20].

Nevertheless, existing investigations of BFSSs have predominantly emphasised foundational mathematical formulations and relatively straightforward applications, with comparatively limited attention devoted to aggregation mechanisms essential for practical implementation. To address this methodological requirement, aggregation operators for BFSS were formulated using t-norm and t-conorm operations [1]. More recently, the incorporation of Frank AOs has further expanded the analytical capabilities of fuzzy-based DM methodologies [14].

1.2 Queries for Research

The present investigation is guided by the following fundamental research questions:

1. How can CFS and BFSSs be integrated within the framework of redefined normal bipolar complex fuzzy soft sets (RNBCFSSs) to improve the accuracy of DM under uncertainty?
2. Which operational laws, score and accuracy functions, and ordering relations are necessary to facilitate the systematic comparison and ranking of redefined normal bipolar complex fuzzy soft numbers (RNBCFSNs) within MCGDM environments involving ambiguity, duality, and uncertainty?
3. How can redefined normal bipolar complex fuzzy soft weighted geometric (RNBCFSWG) and redefined normal bipolar complex fuzzy soft weighted averaging (RNBCFSWA) operators be mathematically formulated for multi-attribute environments, and what fundamental algebraic properties characterise their behaviour?
4. What algorithmic framework can be established within the RNBCFSS environment to ensure robust MCDM when the available information exhibits ambiguity, periodicity, and uncertainty?
5. To what extent can sensitivity analyses involving expert preferences and parameter weights establish the effectiveness and reliability of the proposed RNBCFSS-based MCDM methodology?

6. How can the RNBCFSS framework improve DM for network security solution selection, and what comparative benefits does it provide over conventional approaches when addressing complexity and uncertainty?

1.3 Goals of the Research

The principal objectives of the present investigation are outlined below:

1. To establish a systematic mathematical formulation of RNBCFSS by integrating CFS and BFSS representations, thereby strengthening DM processes under uncertain conditions.
2. To introduce the innovative BCPFSS framework as an extension of the established BPFS model, overcoming the limitations associated with one-dimensional information representation and the resulting loss of information by incorporating a second dimension into MGs to reflect practical decision environments more comprehensively.
3. To formulate the score and accuracy functions, ordering relations, and operational laws required for the systematic evaluation and ranking of BCPFSSs in MADM problems characterised by ambiguity, duality, and uncertainty.
4. To construct specialised aggregation mechanisms, namely RNBCFSWA and RNBCFSWG, for processing complex fuzzy soft set (CFSS) information in multi-attribute environments, while systematically investigating their underlying mathematical characteristics.
5. To develop a dependable MCDM algorithm within the RNBCFSS framework that accommodates ambiguous, periodic, and unreliable information while preserving the accuracy and consistency of decision outcomes.
6. To introduce a novel distance measure for determining criterion weights and expert weights under fuzzy soft aggregation information, providing a practical mathematical mechanism for evaluating complex alternatives and assigning appropriate criterion weights across different DM frameworks.
7. To assess the effectiveness and robustness of the proposed BCPFSS-based MCDM methodology through sensitivity analyses examining variations in parameter weights and expert preferences.
8. To investigate the applicability of the proposed RNBCFSS framework to network security solution selection, demonstrating its comparative advantages over conventional methodologies in representing and managing ambiguity, complexity, and uncertainty.

1.4 Main Contributions

This investigation establishes BCPFSSs as an integrated mathematical framework intended to strengthen MCGDM methodologies in decision environments characterised by complexity and uncertainty. The principal theoretical and practical contributions are summarised as follows:

1. A unified BCPFSS framework is developed by simultaneously incorporating fuzziness, bipolarity through positive and negative membership assessments, parameterisation, and periodicity represented by the amplitude and phase components of complex numbers. Through this integration, the proposed formulation extends conventional CFSs, BFSs, soft sets, and their hybrid structures, providing a comprehensive representation of the multidimensional, dual, and time-dependent characteristics encountered in practical decision problems.
2. A comprehensive mathematical foundation is established through the formal definition of BCPFSNs and the development of their associated set-theoretic operations, algebraic laws, score and accuracy functions, distance measures, and ordering relations. These formulations provide a systematic theoretical basis for subsequent methodological developments and further investigations.

3. Two specialised aggregation mechanisms, BCPFSWA and BCPFSWG, are formulated and examined with respect to their fundamental mathematical properties, particularly idempotency, boundedness, and monotonicity. Building upon these operators, a complete MCDM algorithm is developed within the BCPFSS environment to facilitate structured and consistent evaluations of competing alternatives.
4. The practical applicability of the proposed BCPFSS framework is demonstrated through a real-world network security solution selection problem. Comprehensive comparative and sensitivity analyses are undertaken to evaluate its effectiveness, ranking stability, and robustness relative to existing decision-making methodologies.

1.5 Structure of the Paper

The remainder of this article is organised into six sections. Section 1 establishes the introductory context and outlines the motivation for the investigation. Section 2 reviews the fundamental concepts and previously established mathematical structures underpinning the proposed methodology. Section 3 develops the novel RNBCFSS framework and introduces its principal aggregation mechanisms. Section 4 demonstrates the practical implementation of the proposed MCGDM methodology through a case study involving the evaluation of network security solutions. Section 5 examines the influence of weighted vector parameters and criterion weights, presents comparative and sensitivity analyses, and discusses the methodological advantages of the proposed approach. Finally, Section 6 concludes the investigation by presenting its principal findings and concluding observations.

2. PRELIMINARIES

This section introduces the fundamental mathematical concepts associated with extended FSs and FSSs that provide the theoretical foundation for the subsequent developments. Unless explicitly stated otherwise, image denotes a non-empty parameter set, whereas image represents the universe of discourse.

Definition 2.1 The complex fuzzy set (CFS) $\mathcal{B} = \{(\tau, \mathcal{F}_A(\tau)) : \tau \in \mathcal{X}\}$, where $\mathcal{F}_A : \mathcal{X} tom : m \in \mathcal{B} : |m| < 1\}$ and $\mathcal{F}_A(k) = c + id = \Re_A(\tau) \cdot e^{2\pi i \Im_A(\tau)} A$. Here, $\Re_A(\tau) = \sqrt{c^2 + d^2} \in \mathcal{R}$ & $\Re_A(\tau), \Im_A(\tau) \in [0,1]$, where $i = \sqrt{-1}$, $\Re_A : \mathcal{X} \to [0,1]$ (magnitude), and $\Im_A : \mathcal{X} \to [0,2\pi]$ (phase angle).

Definition 2.2 The bipolar fuzzy set (BFS) of $\mathcal{X}$ is described as follows: $\mathcal{B} = \{\langle \tau, \propto_{\mathcal{B}}^{+}(\tau), \sigma_{\mathcal{B}}^{-}(\tau)\rangle | \tau \in \mathcal{X}\}$, where $\propto_{\mathcal{B}}^{+} : \mathcal{X} \to [0,1]$ and $\sigma_{\mathcal{B}}^{-} : \mathcal{X} \to [-1,0]$ are the positive and negative membership functions, respectively. Bipolar fuzzy numbers (BFNs) are represented by the notation $\langle (\propto_{\mathcal{B}}^{+}, \sigma_{\mathcal{B}}^{-}) \rangle$.

Definition 2.3 Let $\tau = 1,2, \dots, \eta$ be a collection of BFNs, represented as $\gamma_\tau = (\propto_\tau^+, \sigma_\tau^-)$. A BFWA operator is a function $BFWA : \psi^\eta \to \psi$ defined as follows:

$$BFWA(\gamma_1, \gamma_2, \dots, \gamma_\eta) = \oplus_{\tau=1}^{\eta} \varpi_\tau \cdot \gamma_\tau = \left(1 - \prod_{\tau=1}^{\eta} (1 - \propto_\tau^+)^{\varpi_\tau}, -\prod_{\tau=1}^{\eta} (|\sigma_\tau^-|)^{\varpi_\tau}\right).$$

The corresponding weight vector of these BFNs for $\tau = 1,2, \dots, \eta$ is represented here by $\varpi_\tau = [\varpi_1, \varpi_2, \dots, \varpi_\eta]^T$, provided that $\varpi_\tau \in [0,1]$ and the constraint $\sum_{\tau=1}^{\eta} \varpi_\tau = 1$ holds.

Definition 2.4 For $\tau = 1,2, \dots, \eta$, a set of BFN is represented by $\gamma_\tau = (\propto_\tau^+, \sigma_\tau^-)$. A BFWG operator is a function $BFWG : \psi^\eta \to \psi$ defined as follows:

$$BFWG(\gamma_1, \gamma_2, \dots, \gamma_\eta) = \otimes_{\tau=1}^{\eta} \gamma_\tau^{\varpi_\tau} = \left(\prod_{\tau=1}^{\eta} (\propto_\tau^+)^{\varpi_\tau}, -1 + \prod_{\tau=1}^{\eta} (1 + \sigma_\tau^-)^{\varpi_\tau}\right).$$

The corresponding weight vector of these BFNs for $\tau = 1,2,\dots,\eta$ is represented here by $\varpi_\tau = [\varpi_1, \varpi_2, \dots, \varpi_\eta]^T$, provided that $\varpi_\tau \in [0,1]$ and the constraint $\sum_{\tau=1}^{\eta} \varpi_\tau = 1$ holds.

Definition 2.5 A CFBSS over $\mathcal{X}$ is a triplet $(\mathcal{B}, \sigma, \mathcal{A})$, where $\mathcal{B}$ and σ are mappings, provided by $\mathcal{B}: \mathcal{A} \to CFP(\mathcal{X})$ and $\sigma: \mathcal{A} \to CFP(\mathcal{X})$, where $CFP(\mathcal{X})$ is the complex fuzzy power set of $\mathcal{X}$. In other words, $(\mathcal{B}, \tau, \mathcal{A}) = \{(\tau, \mathcal{B}(\epsilon), \sigma(\epsilon)) | \tau \in \mathcal{X}\}$, where, $\mathcal{B}: \mathcal{A} \to \{m_1 : m_1 \in \mathcal{X} : |m_1| < 1\}$, & $\sigma: \mathcal{A} \to \{m_2 : m_2 \in \mathcal{X} : |m_2| < 1\}$, such that $\mathcal{B}(\epsilon) = m_1 = c_1 + id_1$ & $\sigma(\epsilon) = m_2 = c_2 + id_2$, provided that $0 \leqslant |m_1| + |m_2| \leqslant 1$ or $\mathcal{B}(\epsilon) = \pi_{ij}^{+} \cdot e^{2\pi i(\sigma_{ij}^{+})}$ and $\sigma(\epsilon) = \eta_{ij}^{-} \cdot e^{2\pi i(\sigma_{ij}^{-})}$ and satisfying $0 \leqslant \pi_{ij}^{+} + \eta_{ij}^{-} \leqslant 1$. $0 \leqslant \sigma_{ij}^{+} + \sigma_{ij}^{-} \leqslant 2\pi$. Where, $\pi_{ij}^{+}, \eta_{ij}^{-}$ real valued amplitude terms and the phase terms σ_{ij}^{+} & σ_{ij}^{-} for all $\tau \in \mathcal{X}$.

Definition 2.6 Let $\mathcal{B}_{11} = \{\varpi_{11}^{+} \cdot e^{2\pi i(\sigma_{11}^{+})}, \eta_{11}^{-} \cdot e^{2\pi i(\sigma_{11}^{-})}\}$ and $\mathcal{B}_{12} = \{\varpi_{12}^{+} \cdot e^{2\pi i(\sigma_{12}^{+})}, \eta_{12}^{-} \cdot e^{2\pi i(\sigma_{12}^{-})}\}$ be any two (CFBSNs), and let $\varsigma > 0$.

1. $\mathcal{B}_{11} \oplus \mathcal{B}_{12} = \{(\varpi_{11}^{+} + \varpi_{12}^{+} - \varpi_{11}^{+}\varpi_{12}^{+}) \cdot e^{2\pi i(\sigma_{11}^{+} + \sigma_{12}^{+} - \sigma_{11}^{+}\sigma_{12}^{+})}, (\eta_{11}^{-}\eta_{12}^{-}) \cdot e^{2\pi i(\sigma_{11}^{-}\sigma_{12}^{-})}\}$

2. $\mathcal{B}_{11} \otimes \mathcal{B}_{12} = \{(\varpi_{11}^{+}\varpi_{12}^{+}) \cdot e^{2\pi i(\sigma_{11}^{+}\sigma_{12}^{+})}, (\eta_{11}^{-} + \eta_{12}^{-} - \eta_{11}^{-}\eta_{12}^{-}) \cdot e^{2\pi i(\sigma_{11}^{-} + \sigma_{12}^{-} - \sigma_{11}^{-}\sigma_{12}^{-})}\}$

3. $\varsigma \cdot \mathcal{B}_{11} = \{1 - (1 - \varpi_{11}^{+})^{\varsigma} \cdot e^{2\pi i(1-(1-\sigma_{11}^{+})^{\varsigma})}, (\eta_{11}^{-})^{\varsigma} \cdot e^{2\pi i((\sigma_{11}^{-})^{\varsigma})}\}$

4. $\mathcal{B}_{11}^{\varsigma} = \{(\varpi_{11}^{+})^{\varsigma} \cdot e^{2\pi i((\sigma_{11}^{+})^{\varsigma})}, 1 - (1 - \eta_{11}^{-})^{\varsigma} \cdot e^{2\pi i(1-(1-\sigma_{11}^{-})^{\varsigma})}\}$

Definition 2.7 A Pythagorean interval valued fuzzy set (PIVFS) $\mathcal{B} = \{\wp, \langle \widetilde{\Phi^{\mathcal{T}}}(\wp), \widetilde{\Phi^{\mathcal{F}}}(\wp)\rangle | \wp \in \mathcal{X}\}$, $\widetilde{\Phi^{\mathcal{T}}}: \mathcal{X} \to Int([0,1])$ and $\widetilde{\Phi^{\mathcal{F}}}: \mathcal{X} \to Int([0,1])$ referred to the MG and NMG of $\wp \in \mathcal{X}$ to $\mathcal{B}$ respectively, and $0 \leq (\Phi^{\mathcal{T}}(\wp))^2 + (\Phi^{\mathcal{F}}(\wp))^2 \leq 1$. For, $\mathcal{B} = \langle [\Phi^{\mathcal{T}}, \Phi^{\mathcal{T}}], [\Phi^{\mathcal{F}}, \Phi^{\mathcal{F}}] \rangle$ is called a PIVFN. Here $(Int([0,1])$ denotes the collection of all closed subintervals of the real unit interval $([0,1])$.

Definition 2.8 The PIVFN $\mathcal{B} = \langle [a,b], [x,y] \rangle$. Then SV is

$$S(\mathcal{B}) = \frac{1}{2}((a)^2 + (b)^2 - (x)^2 - (y)^2), \qquad S(\mathcal{B}) \in [-1,1]$$

and AV is

$$H(\mathcal{B}) = \frac{1}{2}((a)^2 + (b)^2 + (x)^2 + (y)^2), \qquad H(\mathcal{B}) \in [0,1].$$

3. AOs for RNBCFSS

In this case, ξ_j and ψ_i represent the weight vectors linked to the experts w_i and the parameters e_j, respectively, provided that $\psi_i \in [0,1]$, $\xi_j \in [0,1]$, and $\psi_i > 0, \xi_j > 0, \sum_{i=1}^{m} \psi_i = 1, \sum_{j=1}^{n} \xi_j = 1$. We explain the idea of RNBCFSS by presenting its operations, where the parameters set is crisp and the approximation set is RNBCFSS over $\mathcal{X}$.

Definition 3.1 Let $\mathcal{B}: E \to CBF(\mathcal{X})$ be a function with $\mathcal{B}(\tau) = \bar{0}$ whenever $\tau \notin \mathcal{B}$. Then $\mathcal{B} = \{(\tau, (\beta_{\mathcal{B}}^{+}(\tau), \beta_{\mathcal{B}}^{-}(\tau), u)) : \tau \in E, \mathcal{B}(\tau) \in CBF(\mathcal{X})\}$ is called a RNBCFSS over $\mathcal{X}$. Here, $\mathcal{B}$ is called the bipolar fuzzy approximation function, $(\mathcal{B}(\tau), u)$ is a element of the RNBCFSS $\mathcal{B}$ and $\mathcal{B}(\tau) = \{\langle u, p_{\mathcal{B}(\tau)}^{+}(u) \cdot e^{r_{\mathcal{B}(\tau)}^{+}(u)}, v_{\mathcal{B}(\tau)}^{-}(u) \cdot e^{w_{\mathcal{B}(\tau)}^{-}(u)} \rangle : u \in \mathcal{X}\}$. We will denote by $\mathcal{X}$ the set of all RNBCFSS over $\mathcal{X}$. If $\mathcal{X} = \{u_1, u_2, \dots, u_n\}$, $E = \{\tau_1, \tau_2, \dots, \tau_m\}$ and $\mathcal{B} \subset E$, and $u \in R$, then $\mathcal{B} \in \mathcal{X}$ as the Table 1.

Table 1
Matrix form of the RNBCFSS Data

$\mathcal{B}$	τ_1	τ_m
u_1	$\langle p^+_{\mathcal{B}(\tau_1)}(u_1)\cdot e^{r^+_{\mathcal{B}(\tau_1)}(u_1)}, v^-_{\mathcal{B}(\tau_1)}(u_1)\cdot e^{w^-_{\mathcal{B}(\tau_1)}(u_1)}\rangle$	$\langle p^+_{\mathcal{B}(\tau_m)}(u_1)\cdot e^{r^+_{\mathcal{B}(\tau_m)}(u_1)}, v^-_{\mathcal{B}(\tau_m)}(u_1)\cdot e^{w^-_{\mathcal{B}(\tau_m)}(u_1)}\rangle$
u_2	$\langle p^+_{\mathcal{B}(\tau_1)}(u_2)\cdot e^{r^+_{\mathcal{B}(\tau_1)}(u_2)}, v^-_{\mathcal{B}(\tau_1)}(u_2)\cdot e^{w^-_{\mathcal{B}(\tau_1)}(u_2)}\rangle$	$\langle p^+_{\mathcal{B}(\tau_m)}(u_2)\cdot e^{r^+_{\mathcal{B}(\tau_m)}(u_2)}, v^-_{\mathcal{B}(\tau_m)}(u_2)\cdot e^{w^-_{\mathcal{B}(\tau_m)}(u_2)}\rangle$
⋮	⋮	⋮
u_n	$\langle p^+_{\mathcal{B}(\tau_1)}(u_n)\cdot e^{r^+_{\mathcal{B}(\tau_1)}(u_n)}, v^-_{\mathcal{B}(\tau_1)}(u_n)\cdot e^{w^-_{\mathcal{B}(\tau_1)}(u_n)}\rangle$	$\langle p^+_{\mathcal{B}(\tau_m)}(u_n)\cdot e^{r^+_{\mathcal{B}(\tau_m)}(u_n)}, v^-_{\mathcal{B}(\tau_m)}(u_n)\cdot e^{w^-_{\mathcal{B}(\tau_m)}(u_n)}\rangle$

Definition 3.1 Any two RNBCFSSs $\widetilde{\mathcal{B}}_{11}=\langle(\alpha_1,\beta_1);p^+_{11}\cdot e^{i2\pi r^+_{11}}, v^-_{11}\cdot e^{i2\pi w^-_{11}}\rangle$,

$\widetilde{\mathcal{B}}_{12}=\langle(\alpha_2,\beta_2);p^+_{12}\cdot e^{i2\pi r^+_{12}}, v^-_{12}\cdot e^{i2\pi w^-_{12}}\rangle$ *and* Ψ *is a positive integers. Then*

$$1.\ \widetilde{\mathcal{B}}_{11}\uplus\widetilde{\mathcal{B}}_{12}=\begin{bmatrix}(\alpha_1+\alpha_2,\beta_1+\beta_2);\\ \sqrt[q]{(p^+_{11})^q+(p^+_{12})^q-(p^+_{11})^q\cdot(p^+_{12})^q}\cdot e^{i2\pi\sqrt[q]{\left(\frac{r^+_{11}}{2\pi}\right)^q+\left(\frac{r^+_{12}}{2\pi}\right)^q-\left(\frac{r^+_{11}}{2\pi}\right)^q\cdot\left(\frac{r^+_{12}}{2\pi}\right)^q}},\\ (-|v^-_{11}|\cdot|v^-_{12}|)\cdot e^{-i2\pi\left(\left|\frac{w^-_{11}}{2\pi}\right|\cdot|w^-_{12}|\right)}\end{bmatrix}$$

$$2.\ \widetilde{\mathcal{B}}_{11}\odot\ \widetilde{\mathcal{B}}_{12}=\begin{bmatrix}(\alpha_1\cdot\alpha_2,\beta_1\cdot\beta_2);\\ (p^+_{11}\cdot p^+_{12})\cdot e^{i2\pi(r^+_{11}\cdot r^+_{12})},\\ -\sqrt[q]{|v^-_{11}|^q+|v^-_{12}|^q-|v^-_{11}|^q\cdot|v^-_{12}|^q}\cdot e^{-i2\pi\sqrt[q]{\left|\frac{w^-_{11}}{2\pi}\right|^q+|w^-_{12}|^q-\left|\frac{w^-_{11}}{2\pi}\right|^q\cdot|w^-_{12}|^q}}\end{bmatrix}$$

$$3.\ \Psi\cdot\widetilde{\mathcal{B}_{11}}=\begin{bmatrix}(q\cdot\alpha,q\cdot\beta);\\ \sqrt[q]{|-(|-(p^+_{11})^q)^q}\cdot e^{i2\pi\sqrt[q]{|-\left(|-\left(\frac{r^+_{11}}{2\pi}\right)^q\right)^q}},\\ -|v^-_{11}|^q\cdot e^{-i2\pi\left|\frac{w^-_{11}}{2\pi}\right|^q}\end{bmatrix}$$

$$4.\ \widetilde{\mathcal{B}_{11}}^{\Psi}=\begin{bmatrix}(\alpha^q,\beta^q);(p^+_{11})^q\cdot e^{i2\pi\left(\frac{r^+_{11}}{2\pi}\right)^q},\\ -\sqrt[q]{-|+(|+(v^-_{11})^q)^q}\cdot e^{-i2\pi\sqrt[q]{-|+\left(|+\left(\frac{w^-_{11}}{2\pi}\right)^q\right)^q}},\end{bmatrix}$$

3.1 Weighted Averaging Operator

Definition 3.2 Let $\widetilde{\mathcal{B}}_{ij}=\langle(\alpha_i,\beta_i);p^+_{ij}\cdot e^{i2\pi r^+_{ij}}, v^-_{ij}\cdot e^{i2\pi w^-_{ij}}\rangle$ be the collection of RNBCFSSs, $W=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ be a weight of $\widetilde{\mathcal{B}}_{ij}$ for decision makers A_i for parameters such that and $\lambda_i\geq 0$, $\cup^n_{i=|}\lambda_i=1$. Then RNBCFSWA $\left(\widetilde{\mathcal{B}}_{11},\widetilde{\mathcal{B}}_{21},\ldots,\widetilde{\mathcal{B}}_{nm}\right)=\cup^m_{j=|}\partial_j\left(\cup^n_{i=|}\lambda_i\widetilde{\mathcal{B}}_{ij}\right)$.

Theorem 3.1 Let $\widetilde{\mathcal{B}}_{ij}=\langle(\alpha_i,\beta_i);p^+_{ij}\cdot e^{i2\pi r^+_{ij}}, v^-_{ij}\cdot e^{i2\pi w^-_{ij}}\rangle$ be the collection of RNBCFSSs. Then $RNBCFSWA\left(\widetilde{\mathcal{B}}_{11},\widetilde{\mathcal{B}}_{21},\ldots,\widetilde{\mathcal{B}}_{nm}\right)$

$$= \begin{bmatrix} \left(\bigcup_{i=|}^{n} \lambda_i \alpha_i, \bigcup_{i=|}^{n} \lambda_i \beta_i \right); \\ \sqrt[q]{| - \bigcap_{j=|}^{m} \left(\bigcap_{i=|}^{n} \left(| - \left(p_{ij}^{+} \right)^q \right)^{\lambda_i} \right)^{\partial_j}} \cdot e^{i2\pi \sqrt[q]{| - \cap_{j=|}^{m} \left(\cap_{i=|}^{n} \left(| - \left(\frac{r_{ij}^{+}}{2\pi} \right)^q \right)^{\lambda_i} \right)^{\partial_j}}}, \\ - \bigcap_{j=|}^{m} \left(\bigcap_{i=|}^{n} \left| v_{ij}^{-} \right|^{\lambda_i} \right)^{\partial_j} \cdot e^{-i2\pi \cap_{j=|}^{m} \left(\cap_{i=|}^{n} \left| \frac{w_{ij}^{-}}{2\pi} \right|^{\lambda_i} \right)^{\partial_j}} \end{bmatrix}.$$

Proof. If $n = 2$ and $m = 2$, then

$$RNBCFSWA\left(\widetilde{\mathcal{B}}_{i1}, \widetilde{\mathcal{B}}_{i2}\right) = \partial_1\left(\uplus_{i=|}^{n} \lambda_i \widetilde{\mathcal{B}}_{i1}\right) \uplus \partial_2\left(\uplus_{i=|}^{n} \lambda_i \widetilde{\mathcal{B}}_{i2}\right)$$
$$= \partial_1\left(\lambda_1 \widetilde{\mathcal{B}}_{11} \uplus \lambda_2 \widetilde{\mathcal{B}}_{21}\right) \uplus \partial_2\left(\lambda_1 \widetilde{\mathcal{B}}_{12} \uplus \lambda_2 \widetilde{\mathcal{B}}_{22}\right)$$

where

$$\lambda_1 \widetilde{\mathcal{B}}_{11} = \begin{bmatrix} (\lambda_1 \alpha_1, \lambda_1 \beta_1); \\ \sqrt[q]{| - (| - (p_{11}^{+})^q)^{\lambda_1}} \cdot e^{i2\pi \sqrt[q]{| - \left(| - \left(\frac{r_{11}^{+}}{2\pi} \right)^q \right)^{\lambda_1}}}, \\ - |v_{11}^{-}|^{\lambda_1} \cdot e^{-i2\pi \left| \frac{w_{11}^{-}}{2\pi} \right|^{\lambda_1}} \end{bmatrix}$$

$$\lambda_2 \widetilde{\mathcal{B}}_{21} = \begin{bmatrix} (\lambda_2 \alpha_2, \lambda_2 \beta_2); \\ \sqrt[q]{| - (| - (p_{21}^{+})^q)^{\lambda_2}} \cdot e^{i2\pi \sqrt[q]{| - \left(| - \left(\frac{r_{21}^{+}}{2\pi} \right)^q \right)^{\lambda_2}}}, \\ - |v_{21}^{-}|^{\lambda_2} \cdot e^{-i2\pi \left| \frac{w_{21}^{-}}{2\pi} \right|^{\lambda_2}} \end{bmatrix}$$

Now,

$$\partial_1\left(\lambda_1 \widetilde{\mathcal{B}}_{11} \uplus \lambda_2 \widetilde{\mathcal{B}}_{21}\right) = \partial_1 \left(\begin{bmatrix} (\lambda_1 \alpha_1 + \lambda_2 \alpha_2, \lambda_1 \beta_1 + \lambda_2 \beta_2); \\ \sqrt[q]{(| - (| - (p_{11}^{+})^q)^{\lambda_1}) + (| - (| - (p_{21}^{+})^q)^{\lambda_2}) - (| - (| - (p_{11}^{+})^q)^{\lambda_1}) \cdot (| - (| - (p_{21}^{+})^q)^{\lambda_2})} \cdot \\ e^{i2\pi \sqrt[q]{\left(| - \left(| - \left(\frac{r_{11}^{+}}{2\pi} \right)^q \right)^{\lambda_1} \right) + \left(| - \left(| - \left(\frac{r_{21}^{+}}{2\pi} \right)^q \right)^{\lambda_2} \right) - \left(| - \left(| - \left(\frac{r_{11}^{+}}{2\pi} \right)^q \right)^{\lambda_1} \right) \cdot \left(| - \left(| - \left(\frac{r_{21}^{+}}{2\pi} \right)^q \right)^{\lambda_2} \right)}}, \\ - |v_{11}^{-}|^{\lambda_1} \cdot |v_{21}^{-}|^{\lambda_2} \cdot e^{-i2\pi \left| \frac{w_{11}^{-}}{2\pi} \right|^{\lambda_1} \cdot \left| \frac{w_{21}^{-}}{2\pi} \right|^{\lambda_2}} \end{bmatrix} \right)$$

$$= \partial_1 \left(\begin{bmatrix} (\lambda_1\alpha_1 + \lambda_2\alpha_2, \lambda_1\beta_1 + \lambda_2\beta_2); \\ \sqrt[q]{| - (| - (p_{11}^+)^q)^{\lambda_1} (| - (p_{21}^+)^q)^{\lambda_2}} \cdot e^{i2\pi \sqrt[q]{| - \left(| - \left(\frac{r_{11}^+}{2\pi}\right)^q\right)^{\lambda_1} \left(| - \left(\frac{r_{21}^+}{2\pi}\right)^q\right)^{\lambda_2}}}, \\ -|v_{11}^-|^{\lambda_1} \cdot |v_{21}^-|^{\lambda_2} \cdot e^{-i2\pi \left|\frac{w_{11}^-}{2\pi}\right|^{\lambda_1} \cdot \left|\frac{w_{21}^-}{2\pi}\right|^{\lambda_2}} \end{bmatrix} \right)$$

$$= \partial_1 \left(\begin{bmatrix} \left(\bigcup_{i=|}^{2} \lambda_i\alpha_i, \bigcup_{i=|}^{2} \lambda_i\beta_i \right); \\ \sqrt[q]{| - \bigcap_{i=|}^{2} (| - (p_{i1}^+)^q)^{\lambda_i}} \cdot e^{i2\pi \sqrt[q]{| - \cap_{i=|}^{2} \left(| - (r_{i1}^+)^q\right)^{\lambda_i}}}, \\ -\bigcap_{i=|}^{2} |v_{i1}^-|^{\lambda_i} \cdot e^{-i2\pi \cap_{i=|}^{2} \left|\frac{w_{i1}^-}{2\pi}\right|^{\lambda_i}} \end{bmatrix} \right).$$

$$= \begin{bmatrix} \left(\bigcup_{i=|}^{2} \lambda_i\alpha_i, \bigcup_{i=|}^{2} \lambda_i\beta_i \right); \\ \sqrt[q]{| - \left(\bigcap_{i=|}^{2} (| - (p_{i1}^+)^q)^{\lambda_i} \right)^{\partial_1}} \cdot e^{i2\pi \sqrt[q]{| - \left(\cap_{i=|}^{2} \left(| - (r_{i1}^+)^q\right)^{\lambda_i} \right)^{\partial_1}}}, \\ -\left(\bigcap_{i=|}^{2} |v_{i1}^-|^{\lambda_i} \right)^{\partial_1} \cdot e^{-i2\pi \left(\cap_{i=|}^{2} \left|\frac{w_{i1}^-}{2\pi}\right|^{\lambda_i} \right)^{\partial_1}} \end{bmatrix}.$$

Similarly,

$$\partial_2\left(\lambda_1\widetilde{\mathcal{B}}_{12} \uplus \lambda_2\widetilde{\mathcal{B}}_{22}\right) = \begin{bmatrix} \left(\bigcup_{i=|}^{2} \lambda_i\alpha_i, \bigcup_{i=|}^{2} \lambda_i\beta_i \right); \\ \sqrt[q]{| - \left(\bigcap_{i=|}^{2} (| - (p_{i2}^+)^q)^{\lambda_i} \right)^{\partial_2}} \cdot e^{i2\pi \sqrt[q]{| - \left(\cap_{i=|}^{2} \left(| - (r_{i2}^+)^q\right)^{\lambda_i} \right)^{\partial_2}}}, \\ -\left(\bigcap_{i=|}^{2} |v_{i2}^-|^{\lambda_i} \right)^{\partial_2} \cdot e^{-i2\pi \left(\cap_{i=|}^{2} \left|\frac{w_{i2}^-}{2\pi}\right|^{\lambda_i} \right)^{\partial_2}} \end{bmatrix}.$$

Now, $\partial_1\left(\lambda_1\widetilde{\mathcal{B}}_{11} \uplus \lambda_2\widetilde{\mathcal{B}}_{21}\right) \uplus \partial_2\left(\lambda_1\widetilde{\mathcal{B}}_{12} \uplus \lambda_2\widetilde{\mathcal{B}}_{22}\right) =$

$$\left[\begin{array}{l}(\lambda_1\alpha_1+\lambda_2\alpha_2,\lambda_1\beta_1+\lambda_2\beta_2);\\ \sqrt[q]{\left(|-((|-(p_{11}^{+})^{q})^{\lambda_1})\right)^{\partial_1}+\left(|-((|-(p_{21}^{+})^{q})^{\lambda_2})\right)^{\partial_2}}\\ -\left(|-((|-(p_{11}^{+})^{q})^{\lambda_1})\right)^{\partial_1}\cdot\left(|-((|-(p_{21}^{+})^{q})^{\lambda_2})\right)^{\partial_2},\\ e^{i2\pi\sqrt[q]{\left(|-\left(\left(|-\left(\frac{r_{11}^{+}}{2\pi}\right)^{q}\right)^{\lambda_1}\right)\right)^{\partial_1}+\left(|-\left(\left(|-\left(\frac{r_{21}^{+}}{2\pi}\right)^{q}\right)^{\lambda_2}\right)\right)^{\partial_2}}}\\ -\left(|-\left(\left(|-\left(\frac{r_{11}^{+}}{2\pi}\right)^{q}\right)^{\lambda_1}\right)\right)^{\partial_1}\cdot\left(|-\left(\left(|-\left(\frac{r_{21}^{+}}{2\pi}\right)^{q}\right)^{\lambda_2}\right)\right)^{\partial_2}},\\ -\left(|v_{11}^{-}|^{\lambda_1}\right)^{\partial_1}\cdot\left(|v_{21}^{-}|^{\lambda_2}\right)^{\partial_2}\cdot e^{-i2\pi\left(\left|\frac{w_{11}^{-}}{2\pi}\right|^{\lambda_1}\right)^{\partial_1}\cdot\left(\left|\frac{w_{21}^{-}}{2\pi}\right|^{\lambda_2}\right)^{\partial_2}}\end{array}\right]$$

$$=\left[\begin{array}{l}(\lambda_1\alpha_{11}+\lambda_2\alpha_{21},\lambda_1\beta_{11}+\lambda_2\beta_{21});\\ \sqrt[q]{|-((|-(p_{11}^{+})^{q})^{\lambda_1})^{\partial_1}((|-(p_{21}^{+})^{q})^{\lambda_2})^{\partial_2}}\cdot\\ e^{i2\pi\sqrt[q]{|-\left(\left(|-\left(\frac{r_{11}^{+}}{2\pi}\right)^{q}\right)^{\lambda_1}\right)^{\partial_1}\left(\left(|-\left(\frac{r_{21}^{+}}{2\pi}\right)^{q}\right)^{\lambda_2}\right)^{\partial_2}}},\\ -\left(|v_{11}^{-}|^{\lambda_1}\right)^{\partial_1}\cdot\left(|v_{21}^{-}|^{\lambda_2}\right)^{\partial_2}\cdot e^{-i2\pi\left(\left|\frac{w_{11}^{-}}{2\pi}\right|^{\lambda_1}\right)^{\partial_1}\cdot\left(\left|\frac{w_{21}^{-}}{2\pi}\right|^{\lambda_2}\right)^{\partial_2}}\end{array}\right]\quad RNBCFSWA\left(\widetilde{\mathcal{B}}_{i1},\widetilde{\mathcal{B}}_{i2}\right)=$$

$$\left[\begin{array}{l}\left(\bigcup_{i=|}^{2}\lambda_i\alpha_i,\bigcup_{i=|}^{2}\lambda_i\beta_i\right);\\ \sqrt[q]{|-\bigcap_{j=|}^{2}\left(\bigcap_{i=|}^{2}\left(|-\left(p_{ij}^{+}\right)^{q}\right)^{\lambda_i}\right)^{\partial_j}}\cdot e^{i2\pi\sqrt[q]{|-\cap_{j=|}^{2}\left(\cap_{i=|}^{2}\left(|-\left(\frac{r_{ij}^{+}}{2\pi}\right)^{q}\right)^{\lambda_i}\right)^{\partial_j}}},\\ -\bigcap_{j=|}^{2}\left(\bigcap_{i=|}^{2}\left|v_{ij}^{-}\right|^{\lambda_i}\right)^{\partial_j}\cdot e^{-i2\pi\cap_{j=|}^{2}\left(\cap_{i=|}^{2}\left|\frac{w_{ij}^{-}}{2\pi}\right|^{\lambda_i}\right)^{\partial_j}}\end{array}\right].$$

Generally, $n\geq 3$ and $m\geq 3$. Then, $RNBCFSWA\left(\widetilde{\mathcal{B}}_{i1},\widetilde{\mathcal{B}}_{i2},\ldots,\widetilde{\mathcal{B}}_{il}\right)=$

$$\left[\begin{array}{l}\left(\bigcup_{i=1}^{l}\lambda_i\alpha_i,\bigcup_{i=1}^{l}\lambda_i\beta_i\right);\\ \sqrt[q]{1-\bigcap_{j=1}^{l}\left(\bigcap_{i=1}^{l}\left(1-\left(p_{ij}^{+}\right)^{q}\right)^{\lambda_i}\right)^{\partial_j}}\cdot e^{i2\pi\sqrt[q]{1-\cap_{j=1}^{l}\left(\cap_{i=1}^{l}\left(1-\left(\frac{r_{ij}^{+}}{2\pi}\right)^{q}\right)^{\lambda_i}\right)^{\partial_j}}},\\ -\bigcap_{j=1}^{l}\left(\bigcap_{i=1}^{l}\left|v_{ij}^{-}\right|^{\lambda_i}\right)^{\partial_j}\cdot e^{-i2\pi\cap_{j=1}^{l}\left(\cap_{i=1}^{l}\left|\frac{w_{ij}^{-}}{2\pi}\right|^{\lambda_i}\right)^{\partial_j}}\end{array}\right].$$

If $n=l+1$ and $m=l+1$, then RNBCFSWA $\left(\widetilde{\mathcal{B}}_{i1},\widetilde{\mathcal{B}}_{i2},\ldots,\widetilde{\mathcal{B}}_{il},\widetilde{\mathcal{B}}_{i(l+1)}\right)$=

$$\left[\begin{array}{l}\left(\bigcup_{i=1}^{l}\lambda_i\alpha_i+\lambda_{l+1}\alpha_{l+1},\bigcup_{i=1}^{l}\lambda_i\beta_i+\lambda_{l+1}\beta_{l+1}\right);\\ \sqrt[q]{\bigcup_{j=1}^{l}\left(\bigcup_{i=1}^{l}\left(1-\left(1-\left(p_{ij}^{+}\right)^{q}\right)^{\lambda_i}\right)\right)^{\partial_i}+\left(1-\left(\left(1-\left(p_{l+1}^{+}\right)^{q}\right)^{\lambda_{l+1}}\right)\right)^{\partial_{l+1}}}\\ \quad-\sqrt{\bigcap_{j=1}^{l}\left(\bigcap_{i=1}^{l}\left(1-\left(1-\left(p_{ij}^{+}\right)^{q}\right)^{\lambda_i}\right)\right)^{\partial_i}\cdot\left(1-\left(\left(1-\left(p_{l+1}^{+}\right)^{q}\right)^{\lambda_{l+1}}\right)\right)^{\partial_{l+1}}}\\ e^{i2\pi\sqrt[q]{\cup_{j=1}^{l}\left(\cup_{i=1}^{l}\left(1-\left(1-\left(\frac{r_{ij}^{+}}{2\pi}\right)^{q}\right)^{\lambda_i}\right)\right)^{\partial_i}+\left(1-\left(\left(1-\left(\frac{r_{l+1}^{+}}{2\pi}\right)^{q}\right)^{\lambda_{l+1}}\right)\right)^{\partial_{l+1}}}}\\ \quad-\sqrt{\cap_{j=1}^{l}\left(\cap_{i=1}^{l}\left(1-\left(1-\left(\frac{r_{ij}^{+}}{2\pi}\right)^{q}\right)^{\lambda_i}\right)\right)^{\partial_i}\cdot\left(1-\left(\left(1-\left(\frac{r_{l+1}^{+}}{2\pi}\right)^{q}\right)^{\lambda_{l+1}}\right)\right)^{\partial_{l+1}}}},\\ -\bigcap_{j=1}^{l}\left(\bigcap_{i=1}^{l}\left|v_{ij}^{-}\right|^{\lambda_i}\right)^{\partial_j}\cdot\left(\left|v_{l+1}^{-}\right|^{\lambda_{l+1}}\right)^{\partial_{l+1}}\cdot e^{-i2\pi\cap_{j=1}^{l}\left(\cap_{i=1}^{l}\left|\frac{w_{ij}^{-}}{2\pi}\right|^{\lambda_i}\right)^{\partial_j}\cdot\left(\left|\frac{w_{l+1}^{-}}{2\pi}\right|^{\lambda_{l+1}}\right)^{\partial_{l+1}}}\end{array}\right].$$

$$= \begin{bmatrix} \left(\bigcup_{i=|}^{l+1} \lambda_i \alpha_i, \bigcup_{i=|}^{l+1} \lambda_i \beta_i \right); \\ \sqrt[q]{| - \bigcap_{j=|}^{l+1} \left(\bigcap_{i=|}^{l+1} \left(| - \left(p_{ij}^{+} \right)^q \right)^{\lambda_i} \right)^{\partial_j}} \cdot e^{i2\pi \sqrt[q]{| - \cap_{j=|}^{l+1} \left(\cap_{i=|}^{l+1} \left(| - \left(\frac{r_{ij}^{+}}{2\pi} \right)^q \right)^{\lambda_i} \right)^{\partial_j}}}, \\ - \bigcap_{j=|}^{l+1} \left(\bigcap_{i=|}^{l+1} \left| v_{ij}^{-} \right|^{\lambda_i} \right)^{\partial_j} \cdot e^{-i2\pi \cap_{j=|}^{l+1} \left(\cap_{i=|}^{l+1} \left| \frac{w_{ij}^{-}}{2\pi} \right|^{\lambda_i} \right)^{\partial_j}} \end{bmatrix}$$

Theorem 3.2 If all $\widetilde{\mathcal{B}}_{ij} = \langle (\alpha_i, \beta_i); p_{ij}^{+} \cdot e^{i2\pi r_{ij}^{+}}, v_{ij}^{-} \cdot e^{i2\pi w_{ij}^{-}} \rangle$ are equal, then $\left(\widetilde{\mathcal{B}}_{11}, \widetilde{\mathcal{B}}_{21}, \ldots, \widetilde{\mathcal{B}}_{nm} \right) = \widetilde{\mathcal{B}}$ (idempotency property).

Proof. Given that $(\alpha_i, \beta_i) = (\alpha, \beta)$, $p_{ij}^{+} \cdot e^{i2\pi r_{ij}^{+}} = p^{+} \cdot e^{i2\pi r^{+}}$ and $v_{ij}^{-} \cdot e^{i2\pi w_{ij}^{-}} = v^{-} \cdot e^{i2\pi w^{-}}$ and $\cup_{i=|}^{n} \lambda_i = 1$. Now, $RNBCFSWA\left(\widetilde{\mathcal{B}}_{11}, \widetilde{\mathcal{B}}_{21}, \ldots, \widetilde{\mathcal{B}}_{nm} \right)$

$$= \begin{bmatrix} \left(\bigcup_{i=|}^{n} \lambda_i \alpha_i, \bigcup_{i=|}^{n} \lambda_i \beta_i \right); \\ \sqrt[q]{| - \bigcap_{j=|}^{m} \left(\bigcap_{i=|}^{n} \left(| - \left(p_{ij}^{+} \right)^q \right)^{\lambda_i} \right)^{\partial_j}} \cdot e^{i2\pi \sqrt[q]{| - \cap_{j=|}^{m} \left(\cap_{i=|}^{n} \left(| - \left(\frac{r_{ij}^{+}}{2\pi} \right)^q \right)^{\lambda_i} \right)^{\partial_j}}}, \\ - \bigcap_{j=|}^{m} \left(\bigcap_{i=|}^{n} \left| v_{ij}^{-} \right|^{\lambda_i} \right)^{\partial_j} \cdot e^{-i2\pi \cap_{j=|}^{m} \left(\cap_{i=|}^{n} \left| \frac{w_{ij}^{-}}{2\pi} \right|^{\lambda_i} \right)^{\partial_j}} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\alpha \bigcup_{i=|}^{n} \lambda_i, \beta \bigcup_{i=|}^{n} \lambda_i \right); \\ \sqrt[q]{| - \left(\left(| - \left(p^{+} \right)^q \right)^{\cup_{i=|}^{n} \lambda_i} \right)^{\cup_{j=|}^{m} \partial_j}} \cdot e^{i2\pi \sqrt[q]{| - \left(\left(| - \left(\frac{r^{+}}{2\pi} \right)^q \right)^{\cup_{i=|}^{n} \lambda_i} \right)^{\cup_{j=|}^{m} \partial_j}}}, \\ - \left(\left| v^{-} \right|^{\cup_{i=|}^{n} \lambda_i} \right)^{\cup_{j=|}^{m} \partial_j} \cdot e^{-i2\pi \left(\left| \frac{w^{-}}{2\pi} \right|^{\cup_{i=|}^{n} \lambda_i} \right)^{\cup_{j=|}^{m} \partial_j}} \end{bmatrix}$$

$$= \widetilde{\mathcal{B}}$$

Theorem 3.3 Let $\widetilde{\mathcal{B}}_{ij} = \langle (\alpha_i, \beta_i); p_{ij}^{+} \cdot e^{i2\pi r_{ij}^{+}}, v_{ij}^{-} \cdot e^{i2\pi w_{ij}^{-}} \rangle$ be a collection RNBCFSWA, where $\vec{\alpha} =$

$min_j min_i \alpha_{ij}$, $\overleftarrow{\tilde{\alpha}} = max_j max_i \alpha_{ij}$, $\vec{\beta} = max_j max_i \beta_{ij}$, $\overleftarrow{\tilde{\beta}} = min_j min_i \beta_{ij}$, $\overrightarrow{p^+} = min_j min_i p_{ij}^-$, $\overleftarrow{p^+} = max_j max_i p_{ij}^-$, $\overrightarrow{r^+} = min_j min_i r_{ij}^+$, $\overleftarrow{r^+} = max_j max_i r_{ij}^+$, $\overrightarrow{v^-} = min_j min_i v_{ij}^-$, $\overleftarrow{v^-} = max_j max_i v_{ij}^-$, $\overrightarrow{w^-} = min_j min_i w_{ij}^{F^+}$, $\overleftarrow{w^-} = max_j max_i w_{ij}^{F^+}$, where $1 \le i \le n, 1 \le j \le m$ $j = 1,2,\dots,i_j$. Then, $\langle(\vec{\alpha},\vec{\beta}); \overrightarrow{p^+} \cdot e^{i2\pi\overrightarrow{r^+}}, \overleftarrow{v^-} \cdot e^{i2\pi\overleftarrow{w^-}}\rangle \le RNBCFSWA(\widetilde{\mathcal{B}}_{11}, \widetilde{\mathcal{B}}_{21}, \dots, \widetilde{\mathcal{B}}_{nm}) \le \langle(\overleftarrow{\tilde{\alpha}},\overleftarrow{\tilde{\beta}}); \overleftarrow{p^+} \cdot e^{i2\pi\overleftarrow{r^+}}, \overrightarrow{v^-} \cdot e^{i2\pi\overrightarrow{w^-}}\rangle$(boundedness property).

Proof. Since, $\overrightarrow{p^+} = \min_j \min_i p_{ij}^-$, $\overleftarrow{p^+} = \max_j \max_i p_{ij}^-$

$$\overrightarrow{r^+} = min_j min_i r_{ij}^+, \overleftarrow{r^+} = max_j max_i r_{ij}^+ \; and \overrightarrow{p^+} \le p_{ij}^- \le \overleftarrow{p^+} \; and \overrightarrow{r^+} \le r_{ij}^+ \le \overleftarrow{r^+}.$$

We have,

$$\overrightarrow{p^+} \cdot e^{i2\pi\overrightarrow{r^+}} = \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\overrightarrow{p^+}\right)^q\right)^{\lambda_i}\right)^{\partial_j}} \cdot e^{i2\pi\sqrt[q]{1 - \cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1 - \left(\frac{\overrightarrow{r^+}}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}}$$

$$\le \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(p_{ij}^-\right)^q\right)^{\lambda_i}\right)^{\partial_j}} \cdot e^{i2\pi\sqrt[q]{1 - \cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1 - \left(\frac{r_{ij}^+}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}}$$

$$\le \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\overleftarrow{p^+}\right)^q\right)^{\lambda_i}\right)^{\partial_j}} \cdot e^{i2\pi\sqrt[q]{1 - \cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1 - \left(\frac{\overleftarrow{r^+}}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}}$$

$$= \overleftarrow{p^+} \cdot e^{i2\pi\overleftarrow{r^+}}.$$

Since, $\overrightarrow{v^-} = \min_j \min_i v_{ij}^-$, $\overleftarrow{v^-} = \max_j \max_i v_{ij}^-$

$$\overrightarrow{w^-} = min_j min_i w_{ij}^{F^+}, \overleftarrow{w^-} = max_j max_i w_{ij}^{F^+} \; and$$

$$\overrightarrow{v^-} \le v_{ij}^- \le \overleftarrow{v^-} \; and \overrightarrow{w^-} \le w_{ij}^{F^+} \le \overleftarrow{w^-}.$$

We have,

$$\overrightarrow{v^-} \cdot e^{i2\pi\overrightarrow{w^-}} = \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left|\overrightarrow{v^-}\right|^{\lambda_i}\right)^{\partial_j} \cdot e^{i2\pi\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left|\frac{\overrightarrow{w^-}}{2\pi}\right|^{\lambda_i}\right)^{\partial_j}}$$

$$\le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left|v_{ij}^-\right|^{\lambda_i}\right)^{\partial_j} \cdot e^{i2\pi\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left|\frac{w_{ij}^-}{2\pi}\right|^{\lambda_i}\right)^{\partial_j}}$$

$$\leq \bigcap_{j=|}^{m}\left(\bigcap_{i=|}^{n}|\overleftarrow{v^-}|^{\lambda_i}\right)^{\partial_j}\cdot e^{i2\pi\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left|\frac{\overleftarrow{w^-}}{2\pi}\right|^{\lambda_i}\right)^{\partial_j}}$$

$$=\overleftarrow{v^-}\cdot e^{i2\pi\overleftarrow{w^-}}.$$

Since, $\vec{\alpha}=\min_j\min_i\alpha_{ij}$, $\overleftarrow{\alpha}=\max_j\max_i\alpha_{ij}$, $\vec{\beta}=\max_j\max_i\beta_{ij}$, $\overleftarrow{\beta}=\min_j\min_i\beta_{ij}$ and $\vec{\alpha}\leq\alpha_{ij}\leq\overleftarrow{\alpha}$ and $\overleftarrow{\beta}\leq\beta_{ij}\leq\vec{\beta}$. Hence, $\cup_{i=|}^{n}\lambda_i\vec{\alpha}\leq\cup_{i=|}^{n}\lambda_i\alpha_{ij}\leq\cup_{i=|}^{n}\lambda_i\overleftarrow{\alpha}$ and $\cup_{i=|}^{n}\lambda_i\overleftarrow{\beta}\leq\cup_{i=|}^{n}\lambda_i\beta_{ij}\leq\cup_{i=|}^{n}\lambda_i\vec{\beta}$. Therefore,

$$\frac{\uplus_{i=|}^{n}\lambda_i\vec{\alpha}}{2}\times\left[\frac{\left(\sqrt[q]{|-\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left(|-\left(\overrightarrow{p^+}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2+\left(\sqrt[q]{|-\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left(|-\left(\frac{\overrightarrow{r^+}}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2}{2}\right.$$
$$\left.+1-\frac{\left(\cap_{j=|}^{m}\left(\cap_{i=|}^{n}|\overleftarrow{v^-}|^{\lambda_i}\right)^{\partial_j}\right)^2+\left(\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left|\frac{\overleftarrow{w^-}}{2\pi}\right|^{\lambda_i}\right)^{\partial_j}\right)^2}{2}\right]$$

$$\leq\frac{\uplus_{i=|}^{n}\lambda_i\alpha_{ij}}{2}\times\left[\frac{\left(\sqrt[q]{|-\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left(|-\left(p_{ij}^-\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2+\left(\sqrt[q]{|-\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left(|-\left(\frac{r_{ij}^+}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2}{2}\right.$$
$$\left.+1-\frac{\left(\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left|v_{ij}^-\right|^{\lambda_i}\right)^{\partial_j}\right)^2+\left(\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left|\frac{w_{ij}^-}{2\pi}\right|^{\lambda_i}\right)^{\partial_j}\right)^2}{2}\right]$$

$$\leq\frac{\uplus_{i=|}^{n}\lambda_i\overleftarrow{\alpha}}{2}\times\left[\frac{\left(\sqrt[q]{|-\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left(|-\left(\overleftarrow{p^+}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2+\left(\sqrt[q]{|-\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left(|-\left(\frac{\overleftarrow{r^+}}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2}{2}\right.$$
$$\left.+1-\frac{\left(\cap_{j=|}^{m}\left(\cap_{i=|}^{n}|\overrightarrow{v^-}|^{\lambda_i}\right)^{\partial_j}\right)^2+\left(\cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left|\frac{\overrightarrow{w^-}}{2\pi}\right|^{\lambda_i}\right)^{\partial_j}\right)^2}{2}\right]$$

Hence, $\langle(\vec{\alpha},\vec{\beta});\overrightarrow{p^+}\cdot e^{i2\pi\overrightarrow{r^+}},\overleftarrow{v^-}\cdot e^{i2\pi\overleftarrow{w^-}}\rangle\leq RNBCFSWA(\widetilde{\mathcal{B}}_{11},\widetilde{\mathcal{B}}_{21},\ldots,\widetilde{\mathcal{B}}_{nm})$

$$\leq\langle(\overleftarrow{\alpha},\overleftarrow{\beta});\overleftarrow{p^+}\cdot e^{i2\pi\overleftarrow{r^+}},\overrightarrow{v^-}\cdot e^{i2\pi\overrightarrow{w^-}}\rangle.$$

Theorem 3.4 Suppose $\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_2,\ldots,\widetilde{\mathcal{B}}_n$ and $\widetilde{N}_1,\widetilde{N}_2,\ldots,\widetilde{N}_n$ are two sets of RNBCFSWA and $\widetilde{\mathcal{B}}_{ij}=\langle\left(\alpha_{t_{ij}},\beta_{t_{ij}}\right);p_{t_{ij}}^+\cdot e^{i2\pi r_{t_{ij}}^+},v_{t_{ij}}^-\cdot e^{i2\pi w_{t_{ij}}^-}\rangle$ and $\widetilde{N}_{ij}=\langle\left(\alpha_{h_{ij}},\beta_{h_{ij}}\right);p_{h_{ij}}^+\cdot e^{i2\pi r_{h_{ij}}^+},v_{h_{ij}}^-\cdot e^{i2\pi w_{h_{ij}}^-}\rangle$. For all i, if there is $\alpha_{t_{ij}}\leq\beta_{h_{ij}}$, $\left(p_{t_{ij}}^+\right)^2+\left(\frac{r_{t_{ij}}^+}{2\pi}\right)^2\leq\left(p_{h_{ij}}^+\right)^2+\left(\frac{r_{h_{ij}}^+}{2\pi}\right)^2$ and $\left(v_{t_{ij}}^-\right)^2+\left(w_{t_{ij}}^-\right)^2\geq\left(v_{h_{ij}}^-\right)^2+\left(w_{h_{ij}}^-\right)^2$ or $\widetilde{\mathcal{B}}_{ij}\leq\widetilde{N}_i$. Then $RNBCFSWA(\widetilde{\mathcal{B}}_1,\widetilde{\mathcal{B}}_2,\ldots,\widetilde{\mathcal{B}}_n)\leq RNBCFSWA(\widetilde{N}_1,\widetilde{N}_2,\ldots,\widetilde{N}_n)$ (monotonicity property).

Proof. For all i, $\alpha_{t_{ij}}\leq\beta_{h_{ij}}$. Therefore, $\cup_{i=|}^{n}\alpha_{t_{ij}}\leq\cup_{i=|}^{n}\beta_{h_{ij}}$.

$$For\ all\ i, \left(p_{t_{ij}}^{+}\right)^2 \cdot e^{i2\pi\left(\frac{r_{t_{ij}}^{+}}{2\pi}\right)^2} \leq \left(p_{h_{ij}}^{+}\right)^2 \cdot e^{i2\pi\left(\frac{r_{h_{ij}}^{+}}{2\pi}\right)^2}.$$

$$Therefore, 1-\left(p_{t_{ij}}^{+}\right)^2 \cdot e^{i2\pi\left(1-\left(\frac{r_{t_{ij}}^{+}}{2\pi}\right)^2\right)} \geq 1-\left(p_{h_{ij}}^{+}\right)^2 \cdot e^{i2\pi\left(1-\left(\frac{r_{h_{ij}}^{+}}{2\pi}\right)^2\right)}.$$

$$Hence, \cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(p_{t_{ij}}^{+}\right)^2\right)^{\lambda_i}\right)^{\partial_j} \cdot e^{i2\pi\left(\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(\frac{r_{t_{ij}}^{+}}{2\pi}\right)^2\right)^{\lambda_i}\right)^{\partial_j}\right)}$$

$$\geq \bigcap_{j=1}^{m}\left(\bigcap_{i=1}^{n}\left(1-\left(p_{h_{ij}}^{+}\right)^2\right)^{\lambda_i}\right)^{\partial_j} \cdot e^{i2\pi\left(\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(\frac{r_{h_{ij}}^{+}}{2\pi}\right)^2\right)^{\lambda_i}\right)^{\partial_j}\right)}$$

$$and \sqrt[q]{1-\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(p_{t_{ij}}^{+}\right)^q\right)^{\lambda_i}\right)^{\partial_j}} \cdot e^{i2\pi\left(\sqrt[q]{1-\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(\frac{r_{t_{ij}}^{+}}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)}$$

$$\leq \sqrt[q]{1-\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(p_{h_{ij}}^{+}\right)^q\right)^{\lambda_i}\right)^{\partial_j}} \cdot e^{i2\pi\left(\sqrt[q]{1-\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(\frac{r_{h_{ij}}^{+}}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)}.$$

$$\frac{\cup_{i=1}^{n}\alpha_{tij}}{2} \times \left[\frac{\left(\sqrt[q]{1-\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(p_{t_{ij}}^{+}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2+\left(\sqrt[q]{1-\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(\frac{r_{t_{ij}}^{+}}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2}{2}\right.$$
$$\left.+1-\frac{\left(\cap_{j=1}^{m}\cap_{i=1}^{n}\left|v_{t_{ij}}^{-}\right|\right)^2+\left(\cap_{j=1}^{m}\cap_{i=1}^{n}\left|\frac{w_{hij}^{-}}{2\pi}\right|\right)^2}{2}\right]$$

$$\leq \frac{\cup_{i=1}^{n}\alpha_{h_{ij}}}{2} \times \left[\frac{\left(\sqrt[q]{1-\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(p_{h_{ij}}^{+}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2+\left(\sqrt[q]{1-\cap_{j=1}^{m}\left(\cap_{i=1}^{n}\left(1-\left(\frac{r_{h_{ij}}^{+}}{2\pi}\right)^q\right)^{\lambda_i}\right)^{\partial_j}}\right)^2}{2}\right.$$
$$\left.+1-\frac{\left(\cap_{j=1}^{m}\cap_{i=1}^{n}\left|v_{h_{ij}}^{-}\right|\right)^2+\left(\cap_{j=1}^{m}\cap_{i=1}^{n}\left|\frac{w_{hij}^{-}}{2\pi}\right|\right)^2}{2}\right]$$

Hence, RNBCFSWA $\left(\widetilde{\mathcal{B}}_{11}, \widetilde{\mathcal{B}}_{21}, \ldots, \widetilde{\mathcal{B}}_{nm}\right) \leq$ RNBCFSWA $\left(\widetilde{N}_{11}, \widetilde{N}_{21}, \ldots, \widetilde{N}_{nm}\right)$.

3.2 RNBCFS Weighted Geometric (RNBCFSWG) Operator

Definition 3.3 Let $\widetilde{\mathcal{B}}_{ij} = \langle(\alpha_i, \beta_i); p_{ij}^{+} \cdot e^{i2\pi r_{ij}^{+}}, v_{ij}^{-} \cdot e^{i2\pi w_{ij}^{-}}\rangle$ be the collection of RNBCFSSs. Then RNBCFSWG $\left(\widetilde{\mathcal{B}}_{11}, \widetilde{\mathcal{B}}_{21}, \ldots, \widetilde{\mathcal{B}}_{nm}\right) = \cap_{j=|}^{m} \left(\cap_{i=|}^{n} \widetilde{\mathcal{B}}_{ij}^{\lambda_i}\right)^{\partial_j}$.

Corollary 3.1 Let $\widetilde{\mathcal{B}}_{ij} = \langle(\alpha_i, \beta_i); p_{ij}^{+} \cdot e^{i2\pi r_{ij}^{+}}, v_{ij}^{-} \cdot e^{i2\pi w_{ij}^{-}}\rangle$ be the collection of RNBCFSSs. Then RNBCFSWG$\left(\widetilde{\mathcal{B}}_{11}, \widetilde{\mathcal{B}}_{21}, \ldots, \widetilde{\mathcal{B}}_{nm}\right) =$

$$\left[\begin{array}{l} \left(\cap_{i=|}^{n} \alpha_i^{\lambda_i}, \cap_{i=|}^{n} \beta_i^{\lambda_i}\right); \\ \\ \cap_{j=|}^{m} \left(\cap_{i=|}^{n} \left(p_{ij}^{+}\right)^{\lambda_i}\right)^{\partial_j} \cdot e^{i2\pi \cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left(\frac{r_{ij}^{+}}{2\pi}\right)^{\lambda_i}\right)^{\partial_j}}, \\ \\ -\sqrt[q]{-| + \cap_{j=|}^{m} \left(\cap_{i=|}^{n} \left(| + \left|v_{ij}^{-}\right|^{q}\right)^{\lambda_i}\right)^{\partial_j}} \cdot e^{-i2\pi \sqrt[q]{-| + \cap_{j=|}^{m}\left(\cap_{i=|}^{n}\left(| + \left|\frac{w_{ij}^{-}}{2\pi}\right|^{q}\right)^{\lambda_i}\right)^{\partial_j}}} \end{array}\right]$$

Proof. The proof is given by Theorem 3.1.

Corollary 3.2 If all $\widetilde{\mathcal{B}}_{ij} = \langle(\alpha_i, \beta_i); p_{ij}^{+} \cdot e^{i2\pi r_{ij}^{+}}, v_{ij}^{-} \cdot e^{i2\pi w_{ij}^{-}}\rangle$ are equal, then RNBCFSWG$\left(\widetilde{\mathcal{B}}_{11}, \widetilde{\mathcal{B}}_{21}, \ldots, \widetilde{\mathcal{B}}_{nm}\right) = \widetilde{\mathcal{B}}$.

Proof. The proof is given by Theorem 3.2.

Corollary 3.3 The RNBCFSWG operator satisfies both the monotonicity and boundedness properties.

Proof. The proof is given by Theorems 3.3 and 3.4.

4. MADM Approach under RNBCFSSs

Let $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$ and $C = "C_1, C_2, \ldots, C_m\}$ be the sets of n-alternative and m-attribute, respectively. Let $\widetilde{\mathcal{B}}_{ij} = \langle(\alpha_i, \beta_i); p_{ij}^{+} \cdot e^{i2\pi r_{ij}^{+}}, v_{ij}^{-} \cdot e^{i2\pi w_{ij}^{-}}\rangle$ be the RNBCFSSs of alternative $\mathcal{B}_i$ in attribute C_j, where $\mathcal{S} = \left(\widetilde{\mathcal{B}}_{ij}\right)_{n\times m}$ is the matrix representation. The proposed model employs a systematic algorithmic framework that initially formulates the DM problem and identifies the relevant evaluation criteria and feasible alternatives. Subsequently, the decision matrix is constructed and normalised, after which the proposed AOs are applied to aggregate the available information. The resulting evaluations are then processed through the computation of score and accuracy values, enabling the systematic ranking of competing alternatives. This structured procedure demonstrates the integration of the proposed aggregation mechanisms within the MADM framework under RNBCFSS information.

4.1 Algorithm

Step 1: The collection of options is $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$, the set of parameters/criteria is $E = \{e_1, e_2, \ldots, u_c\}$, and the panel of expert decision-makers is $D = \{d_1, d_2, \ldots, d_k\}$. Let $w = (w_1, w_2, \ldots, w_c)^T$ be the criteria weight vector ($\sum w_j = 1$), and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T$ be the expert weight vector ($\sum \lambda_p = 1$).

Step 2: Obtain quantitative or qualitative assessments from every expert $d_p \in D$. RNBCFSNs are used to represent assessments:

$$\mathcal{K}_{ij}^{(p)} = \left(\mu_{ij}^{+(p)} e^{i\theta_{ij}^{+(p)}}, \mu_{ij}^{-(p)} e^{i\theta_{ij}^{-(p)}}\right), \mu_{ij}^{+(p)} \in [0,1], \theta_{ij}^{+(p)} \in [0,2\pi], \mu_{ij}^{-(p)} \in [-1,0], \theta_{ij}^{-(p)} \in [-2\pi, 0]$$

Step 3: Create a k expert average decision matrix $\mathcal{S} = (\widetilde{\mathcal{B}}_{ij})_{n\times m}$ from the individual evaluations.

Step 4: Use the RNBCFSS technique to determine objective weights if criteria weights w_j or expert weights λ_p are unknown. An objective way to calculate weights is not provided by enforcing the normalization condition throughout this article, such as the total of the weights equals one.

Step 5: Find the normalized decision matrices $\mathcal{S} = (\widetilde{\mathcal{B}}_{ij})_{n\times m}$ is normalized into $\overleftarrow{\mathcal{S}} = (\mathcal{B}_{ij})_{n\times m}$

$$\mathcal{B}_{ij} = \langle(\overleftarrow{\alpha_{ij}}, \overleftarrow{\beta_{ij}}); [\overleftarrow{p_{ij}^{+}}, \overleftarrow{r_{ij}^{+}}], [\overleftarrow{v_{ij}^{-}}, \overleftarrow{w_{ij}^{-}}]\rangle$$

is aggregated into

$$\mathcal{B}_{i} = \langle(\overleftarrow{\alpha_{i}}, \overleftarrow{\beta_{i}}); [\overleftarrow{p_{ij}^{+}}, \overleftarrow{r_{ij}^{+}}], [\overleftarrow{v_{ij}^{-}}, \overleftarrow{w_{ij}^{-}}]\rangle$$

where

$$\overleftarrow{\alpha_{ij}} = \frac{\alpha_{ij}}{\max\limits_{i}(\alpha_{ij})}, \overleftarrow{\beta_{ij}} = \frac{\beta_{ij}}{\max\limits_{i}(\beta_{ij})} \cdot \frac{\beta_{ij}}{\alpha_{ij}}, \qquad \overleftarrow{p_{ij}^{+}} = p_{ij}^{+}, \overleftarrow{r_{ij}^{+}} = r_{ij}^{+}$$

Step 6: Use the RNBCFSWA/RNBCFSWG Operator: Using the expert weight vector λ, combine individual expert matrices into a collective RNBCFSS decision matrix

$$\mathcal{S} = (\widetilde{\mathcal{B}}_{ij})_{n\times m}: \mathcal{B}_{ij} = RNBCFSWA_{\lambda}\left(\widetilde{\mathcal{B}}_{ij}^{(1)}, \widetilde{\mathcal{B}}_{ij}^{(2)}, \ldots, \widetilde{\mathcal{B}}_{ij}^{(k)}\right).$$

Then, for each option u_i, aggregate across criteria using weight vector w to generate the overall evaluation vector $\mathcal{A}_i = \mathrm{RNBCFSWA}_w(\mathcal{S}_{i1}, \ldots, \mathcal{S}_{ic})$.

Step 7: At this stage, the evaluation outcomes obtained through the computational procedures implemented in Step-6 are consolidated to determine a final rating for each alternative. The individual scores generated by the previously applied assessment methods are systematically aggregated, providing an integrated basis for comparing the available options. These consolidated ratings reflect the overall performance of each alternative and facilitate a more informed, consistent, and rational DM process. The score value (SV) is

$$S_1(\mathcal{B}) = \frac{\alpha}{2}\left(\frac{(p)^2 + (r)^2}{2} + 1 - \frac{(v)^2 + (w)^2}{2}\right)$$

and

$$S_2(\mathcal{B}) = \frac{\beta}{2}\left(\frac{(p)^2 + (r)^2}{2} + 1 - \frac{(v)^2 + (w)^2}{2}\right)$$

$$\frac{S_1(\mathcal{B}) + S_2(\mathcal{B})}{2}, S(\mathcal{B}) \in [-1,1].$$

The accuracy value (AV) is

$$H_1(\mathcal{B}) = \frac{\alpha}{2}\left(\frac{(p)^2 + (r)^2}{2} + \frac{(v)^2 + (w)^2}{2}\right)$$

and

$$H_2(\mathcal{B}) = \frac{\beta}{2}\left(\frac{(p)^2 + (r)^2}{2} + \frac{(v)^2 + (w)^2}{2}\right)$$

$$\frac{H_1(\mathcal{B}) + H_2(\mathcal{B})}{2}, H(\mathcal{B}) \in [0,1].$$

Step 8: To calculate the relative proximity coefficient C_i, compute the distance of each option $\mathcal{A}_i$ from the ideal positive solution ($\mathcal{A}^+$) and the ideal negative solution ($\mathcal{A}^-$) and $C_i = \frac{d(\mathcal{A}_i,\mathcal{A}^-)}{d(\mathcal{A}_i,\mathcal{A}^+)+d(\mathcal{A}_i,\mathcal{A}^-)}$

Step 9: The options $u_i \in U$ are sorted in decreasing (increasing) order according to $S(\mathcal{A}_i)$ (or C_i). The final rankings should be $H(\mathcal{A}_a)$ if $S(\mathcal{A}_a) = S(\mathcal{A}_b)$. The most valuable option needs to be the greatest one. The last step uses the assessment ratings established in Step-8 to rank the various solutions. This rating, which provides a brief overview of their comparative performance based on the predefined features and evaluation process, highlights the best options. Finding the greatest solutions is made easier with its help.

Step 10: To evaluate the stability and robustness of the resulting alternative rankings, systematically modify the parameter weight vectors image under different scenarios or adjust the operator parameter configurations. The final aggregated ratings provide a comprehensive assessment of each alternative by incorporating the relevant evaluation criteria and the judgements of participating decision-makers. Arranging these ratings in descending order establishes a preference hierarchy, with the highest-ranked alternative representing the most favourable option. Beyond identifying the preferred solution, the consolidated ratings provide an integrated assessment of the relative performance of all available alternatives across the considered parameters. This systematic evaluation facilitates transparent, evidence-based DM and enables decision-makers to select the most appropriate alternative through a comprehensive comparison of the resulting performance rankings.

4.2 An Analysis of the Suggested MCGDM Method in Network Security Solutions

Network security encompasses the protective mechanisms, policies, and technologies implemented to safeguard communication networks and their associated information against cyberattacks, unauthorised access, and misuse. Its principal purpose is to preserve the confidentiality, availability, and reliability of networked systems across diverse digital environments. Effective network protection relies on a multilayered security architecture in which complementary defensive mechanisms regulate access at network boundaries and throughout the internal infrastructure. Each protective layer addresses specific vulnerabilities, mitigates potential threats, and restricts access to authorised individuals and entities. The underlying principle involves establishing successive security controls that collectively protect substantial volumes of information by enforcing appropriate restrictions and verifying authorisation before permitting access to network resources or operations. Different categories of network security solutions

1. Intrusion Prevention Systems (IPS)(L): Intrusion prevention systems (IPS) are designed to identify and block recognised and potentially malicious threats before they compromise core network infrastructure or peripheral devices. These systems may also provide virtual patching capabilities to mitigate network vulnerabilities, alongside deep packet inspection of east–west and north–south traffic, including encrypted communications where inspection is enabled. By continuously analysing network activity, IPS technologies enable organisations to detect anomalous behaviour and identify attack signatures promptly. Upon recognising potentially harmful traffic, the system can automatically interrupt malicious communications while notifying security administrators to facilitate further investigation. Advanced artificial intelligence (AI) algorithms may further strengthen these capabilities by processing substantial volumes of information collected across multiple network nodes and technological layers. Such analytical mechanisms can operate in conjunction with complementary security controls, including standalone intrusion detection systems (IDS), conventional firewalls, and antivirus software, thereby supporting a more comprehensive and coordinated approach to network threat prevention.

2. Antivirus and Sand Boxing(M): Antivirus software and sandboxing technologies constitute essential components of malware detection and prevention strategies, enabling the identification and assessment of potentially malicious files. Whereas antivirus applications primarily identify and mitigate recognised malware threats, sandboxing provides an isolated computational environment for examining suspicious content without exposing the underlying system to potential harm. For example, when an email attachment is opened and a file is downloaded, antivirus software examines the content for established malicious signatures and behavioural indicators. If the analysis identifies a security threat, the affected file is either removed or quarantined to prevent further compromise. Conversely, when the characteristics of a file remain uncertain, sandboxing enables its execution and behavioural examination within a protected environment to determine whether it exhibits malicious properties. By integrating these complementary capabilities with AI-based analytical mechanisms, certain cybersecurity providers can identify previously unobserved threats in less than a second. Fundamentally, sandboxing represents a controlled security procedure through which potentially harmful files or applications are executed, monitored, and analysed under isolated conditions. This environment functions as a digital quarantine mechanism, separating suspicious processes from operational systems and restricting their capacity to compromise other resources, even when malicious behaviour is subsequently confirmed.
3. Cyber Asset Attack Surface Management(N): Comprehensive visibility across organisational attack surfaces is essential for identifying potential vulnerabilities and strengthening cybersecurity resilience. Automated asset discovery technologies enable organisations to identify and assess information technology (IT), operational technology (OT), and Internet of Things (IoT) resources distributed throughout their network environments. These solutions also facilitate the examination of existing security architectures and protective controls to detect configuration errors, inappropriate settings, and other weaknesses that may compromise the overall security posture.
4. Within this context, Cyber Asset Attack Surface Management (CAASM) provides an integrated cybersecurity framework designed to enhance organisational visibility and control over digital assets. It enables IT and security professionals to systematically discover, evaluate, and manage technological resources across cloud-based, hybrid, and on-premises infrastructures. Unlike conventional asset management approaches, which predominantly maintain static inventories, CAASM incorporates automated discovery, continuous monitoring, and risk-based assessment capabilities to support more dynamic security management. By identifying previously unrecognised assets, exposing security gaps, and facilitating timely corrective measures, this approach contributes to improved attack surface governance and greater organisational resilience against evolving cyber threats.
5. Remote Access VPNs(P): Secure connectivity for geographically dispersed users constitutes a fundamental requirement for protecting organisational networks while maintaining uninterrupted access to essential digital resources. Remote access virtual private networks (VPNs) address this requirement by establishing encrypted communication channels over public internet connections, including potentially unsecured Wi-Fi networks, thereby enabling employees to access organisational systems securely from personal devices and remote locations. These technologies are particularly valuable within hybrid working arrangements, where personnel require reliable connectivity without exposing sensitive information to unauthorised interception.

   A remote access VPN enables an individual to establish a protected connection to a private

network from an external location, making it suitable for employees working outside organisational premises and users accessing private resources through public networks. Different remote access VPN implementations employ distinct communication protocols to encrypt and tunnel information transmitted across the internet, thereby protecting data confidentiality and restricting unauthorised network access. Following successful authentication and connection establishment, users can access the network resources permitted by their assigned privileges, potentially obtaining functionality comparable to that available through an on-site connection. Consequently, remote access VPN technologies support operational flexibility, secure information exchange, and workforce productivity while strengthening the protection of organisational communication infrastructure.

Network Access Control (NAC)(Q): Network access control (NAC) provides a systematic security mechanism for regulating connectivity to organisational networks by ensuring that access is restricted to authorised users and devices satisfying predefined compliance requirements. Functioning as a protective access gateway, NAC solutions identify and authenticate connecting devices, evaluate their adherence to established security policies, and permit network connectivity only when the relevant conditions are fulfilled. This approach enables organisations to prevent potentially vulnerable personal devices from accessing sensitive infrastructure while supporting the secure management of OT and IoT deployments. By integrating device identification, authentication, and compliance verification, NAC strengthens organisational control over network participation and reduces exposure to unauthorised access and associated cybersecurity threats. Fundamentally, NAC establishes a policy-driven admission framework through which individuals and devices are granted access to private or corporate networks only after satisfying the prescribed security and authorisation standards.

4.2 Definition of Assessment Criteria

The evaluation of network security solutions requires a structured set of assessment criteria encompassing relevant security characteristics and their associated sub-attributes. Establishing these criteria provides a consistent analytical foundation for comparing alternative security mechanisms and supports the adaptability of MCDM methodologies within network security environments. Network security evaluation encompasses four principal benefits and challenges, while the five most significant benefits are outlined as follows:

Protecting Sensitive Data & Preventing Cyber Threats (τ_1): Protecting sensitive information constitutes a fundamental objective of network security, particularly in safeguarding financial records, customer information, intellectual property, and personal data against ransomware, phishing, malware, and other cyber threats. Such information represents a valuable target for cybercriminals, who frequently attempt to compromise banking details, consumer records, and confidential organisational assets. Successful data breaches can generate substantial financial losses, reputational damage, and legal consequences, highlighting the importance of establishing comprehensive information protection mechanisms.

Organisations can mitigate these risks and strengthen regulatory compliance by implementing effective data security policies and appropriate technical safeguards. Across industries, the preservation of sensitive information remains a critical organisational priority, particularly as cyberattacks become increasingly sophisticated and difficult to detect. Consequently, robust data protection strategies are essential not only for preventing unauthorised disclosure and exploitation but also for maintaining stakeholder confidence and institutional credibility. A comprehensive approach to information security may therefore incorporate complementary protective measures, ranging from advanced encryption technologies and access restrictions to employee awareness

programmes and cybersecurity training, collectively strengthening organisational resilience against evolving digital threats.

Ensuring Business Continuity & System Resilience (τ_2): Effective network security strengthens organisational resilience by enabling businesses to withstand cyber threats, maintain essential operations, and minimise service disruptions and associated downtime. Ensuring business continuity requires a systematically designed framework capable of supporting operational stability during unexpected incidents, including cyberattacks, power failures, and natural disasters. Within this context, system resilience refers to the ability of software, organisational, or engineering systems to tolerate disturbances, operational failures, and adverse conditions while preserving their essential functions.

From a system design perspective, resilience encompasses both the capacity to withstand disruptive events and the ability to restore normal operations promptly following failures or external stresses. A resilient system therefore incorporates preventive, adaptive, and recovery mechanisms that limit functional degradation, reduce interruptions, and facilitate the timely restoration of services. By integrating these capabilities into network security architectures, organisations can strengthen operational continuity, improve their responsiveness to unforeseen disruptions, and maintain critical functionality with minimal performance deterioration or downtime.

Regulatory Compliance & Risk Management (τ_3): Maintaining effective network security requires organisations to comply with established regulatory and industry frameworks, including the General Data Protection Regulation (GDPR) and the Payment Card Industry Data Security Standard (PCI DSS), thereby reducing their exposure to financial penalties and regulatory sanctions. Within organisational governance, risk management and compliance represent closely interconnected functions that collectively protect businesses against legal liabilities, reputational deterioration, and financial consequences arising from regulatory violations. These functions constitute essential components of the broader governance, risk, and compliance (GRC) framework, which integrates organisational oversight, risk mitigation, and regulatory accountability into a coordinated management structure. Compliance establishes the legal, regulatory, and industry-specific requirements that organisations must satisfy, providing a systematic foundation for maintaining operational integrity and adherence to applicable standards.

Strengthening Access Control & Authentication (τ_4): Network security strengthens the protection of critical information by ensuring that access is granted exclusively to authorised individuals through effective authentication and access control mechanisms, thereby reducing the likelihood of insider threats and unauthorised activities. Access control constitutes a fundamental cybersecurity process that regulates interactions with networks, information systems, and digital resources according to predefined security policies. It determines the individuals or entities permitted to access specific resources and defines the operations they are authorised to perform. To enforce these restrictions, access control systems incorporate identity verification mechanisms that authenticate users through established credentials, including passwords, personal identification numbers (PINs), biometric authentication, and security tokens. By integrating authentication procedures with policy-based authorisation, these mechanisms help prevent unauthorised access and strengthen the confidentiality and integrity of organisational information assets.

Figure 1 illustrates the flowchart of network security solutions.

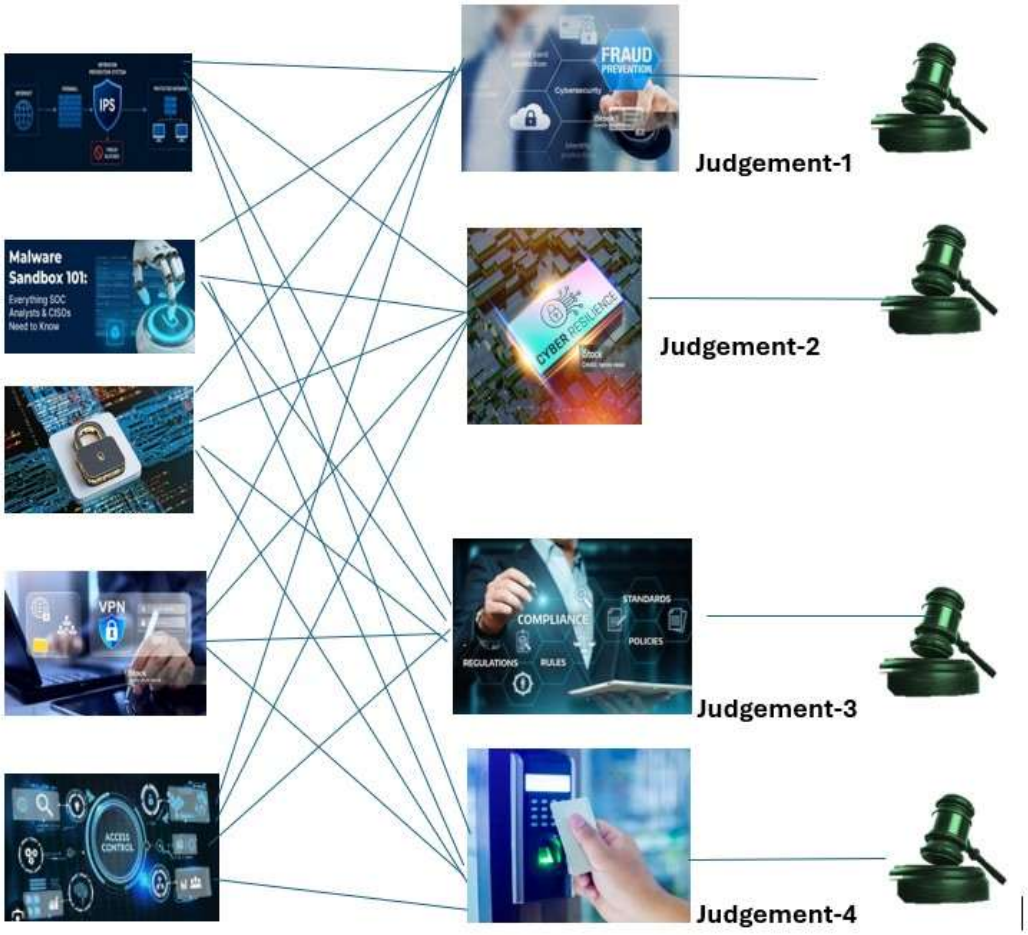


Fig.1: Flow Chart of the Network Security Solutions

Five distinct network security systems were randomly selected as potential alternatives for evaluation. The selection process incorporates four assessment criteria to establish the weighted parameters governing the comparative analysis of the available security solutions. Accordingly, four evaluation judgements are considered, with corresponding weights of 0.35, 0.3, 0.25, and 0.1, respectively. The primary objective is to identify the most suitable alternative by systematically assessing the extent to which each security system satisfies the specified requirements. The decision values employed in this evaluation are presented in Tables 2, 3, 4, and 5.

Table 2
Expert-1 Decision Values

$Option$	τ_1	τ_2	τ_3	τ_4
L	$[0.65 \cdot e^{0.7}, -0.55 \cdot e^{0.45}]$	$[0.65 \cdot e^{0.7}, -0.5 \cdot e^{0.55}]$	$[0.65 \cdot e^{0.65}, -0.45 \cdot e^{0.55}]$	$[0.7 \cdot e^{0.75}, -0.55 \cdot e^{0.45}]$
M	$[0.5 \cdot e^{0.55}, -0.5 \cdot e^{0.6}]$	$[0.6 \cdot e^{0.45}, -0.55 \cdot e^{0.6}]$	$[0.6 \cdot e^{0.45}, -0.55 \cdot e^{0.4}]$	$[0.55 \cdot e^{0.5}, -0.45 \cdot e^{0.5}]$
N	$[0.6 \cdot e^{0.45}, -0.6 \cdot e^{0.55}]$	$[0.45 \cdot e^{0.55}, -0.6 \cdot e^{0.55}]$	$[0.65 \cdot e^{0.6}, -0.6 \cdot e^{0.3}]$	$[0.7 \cdot e^{0.65}, -0.5 \cdot e^{0.6}]$
P	$[0.5 \cdot e^{0.55}, -0.45 \cdot e^{0.55}]$	$[0.6 \cdot e^{0.5}, -0.45 \cdot e^{0.55}]$	$[0.45 \cdot e^{0.55}, -0.55 \cdot e^{0.55}]$	$[0.5 \cdot e^{0.6}, -0.6 \cdot e^{0.65}]$
Q	$[0.45 \cdot e^{0.6}, -0.6 \cdot e^{0.65}]$	$[0.65 \cdot e^{0.6}, -0.55 \cdot e^{0.6}]$	$[0.6 \cdot e^{0.45}, -0.45 \cdot e^{0.6}]$	$[0.55 \cdot e^{0.65}, -0.65 \cdot e^{0.5}]$

Table 3
Expert-2 Decision Values

$Option$	τ_1	τ_2	τ_3	τ_4
L	$[0.75 \cdot e^{0.6}, -0.55 \cdot e^{-0.6}]$	$[0.6 \cdot e^{0.7}, -0.5 \cdot e^{-0.6}]$	$[0.55 \cdot e^{0.65}, -0.35 \cdot e^{-0.65}]$	$[0.65 \cdot e^{0.55}, -0.55 \cdot e^{-0.55}]$
M	$[0.45 \cdot e^{0.5}, -0.45 \cdot e^{-0.5}]$	$[0.65 \cdot e^{0.6}, -0.65 \cdot e^{-0.7}]$	$[0.45 \cdot e^{0.65}, -0.55 \cdot e^{-0.7}]$	$[0.6 \cdot e^{0.45}, -0.5 \cdot e^{-0.65}]$
N	$[0.6 \cdot e^{0.65}, -0.65 \cdot e^{-0.6}]$	$[0.55 \cdot e^{0.5}, -0.45 \cdot e^{-0.55}]$	$[0.4 \cdot e^{0.55}, -0.55 \cdot e^{-0.65}]$	$[0.7 \cdot e^{0.65}, -0.7 \cdot e^{-0.5}]$
P	$[0.5 \cdot e^{0.45}, -0.7 \cdot e^{-0.55}]$	$[0.45 \cdot e^{0.6}, -0.5 \cdot e^{-0.65}]$	$[0.3 \cdot e^{0.45}, -0.65 \cdot e^{-0.6}]$	$[0.6 \cdot e^{0.6}, -0.65 \cdot e^{-0.45}]$
Q	$[0.55 \cdot e^{0.6}, -0.55 \cdot e^{-0.45}]$	$[0.5 \cdot e^{0.55}, -0.55 \cdot e^{-0.65}]$	$[0.35 \cdot e^{0.6}, -0.6 \cdot e^{-0.55}]$	$[0.55 \cdot e^{0.5}, -0.6 \cdot e^{-0.35}]$

Table 4
Expert-3 Decision Values

$Option$	τ_1	τ_2	τ_3	τ_4
L	$[0.7 \cdot e^{0.65}, -0.5 \cdot e^{-0.65}]$	$[0.65 \cdot e^{0.7}, -0.55 \cdot e^{-0.6}]$	$[0.7 \cdot e^{0.65}, -0.45 \cdot e^{-0.6}]$	$[0.55 \cdot e^{0.65}, -0.45 \cdot e^{-0.5}]$
M	$[0.6 \cdot e^{0.55}, -0.65 \cdot e^{-0.6}]$	$[0.5 \cdot e^{0.45}, -0.5 \cdot e^{-0.55}]$	$[0.65 \cdot e^{0.5}, -0.55 \cdot e^{-0.7}]$	$[0.6 \cdot e^{0.65}, -0.5 \cdot e^{-0.6}]$
N	$[0.55 \cdot e^{0.5}, -0.45 \cdot e^{-0.55}]$	$[0.45 \cdot e^{0.6}, -0.6 \cdot e^{-0.65}]$	$[0.6 \cdot e^{0.45}, -0.65 \cdot e^{-0.65}]$	$[0.65 \cdot e^{0.7}, -0.65 \cdot e^{-0.45}]$
P	$[0.45 \cdot e^{0.5}, -0.5 \cdot e^{-0.6}]$	$[0.55 \cdot e^{0.65}, -0.45 \cdot e^{-0.55}]$	$[0.55 \cdot e^{0.6}, -0.5 \cdot e^{-0.6}]$	$[0.7 \cdot e^{0.65}, -0.35 \cdot e^{-0.55}]$
Q	$[0.65 \cdot e^{0.6}, -0.55 \cdot e^{-0.65}]$	$[0.6 \cdot e^{0.55}, -0.35 \cdot e^{-0.4}]$	$[0.5 \cdot e^{0.45}, -0.6 \cdot e^{-0.55}]$	$[0.75 \cdot e^{0.55}, -0.45 \cdot e^{-0.5}]$

Table 5
Expert-4 Decision Values

$Option$	τ_1	τ_2	τ_3	τ_4
L	$[0.7 \cdot e^{0.5}, -0.55 \cdot e^{-0.6}]$	$[0.7 \cdot e^{0.65}, -0.55 \cdot e^{-0.5}]$	$[0.6 \cdot e^{0.65}, -0.45 \cdot e^{-0.45}]$	$[0.55 \cdot e^{0.5}, -0.55 \cdot e^{-0.5}]$
M	$[0.5 \cdot e^{0.55}, -0.6 \cdot e^{-0.65}]$	$[0.55 \cdot e^{0.5}, -0.45 \cdot e^{-0.5}]$	$[0.65 \cdot e^{0.6}, -0.45 \cdot e^{-0.5}]$	$[0.45 \cdot e^{0.65}, -0.5 \cdot e^{-0.6}]$
N	$[0.55 \cdot e^{0.6}, -0.65 \cdot e^{-0.6}]$	$[0.6 \cdot e^{0.65}, -0.55 \cdot e^{-0.6}]$	$[0.45 \cdot e^{0.6}, -0.35 \cdot e^{-0.6}]$	$[0.5 \cdot e^{0.5}, -0.6 \cdot e^{-0.65}]$
P	$[0.6 \cdot e^{0.65}, -0.6 \cdot e^{-0.55}]$	$[0.45 \cdot e^{0.55}, -0.6 \cdot e^{-0.65}]$	$[0.5 \cdot e^{0.45}, -0.5 \cdot e^{-0.55}]$	$[0.6 \cdot e^{0.45}, -0.55 \cdot e^{-0.55}]$
Q	$[0.65 \cdot e^{0.55}, -0.5 \cdot e^{-0.65}]$	$[0.6 \cdot e^{0.55}, -0.55 \cdot e^{-0.5}]$	$[0.6 \cdot e^{0.65}, -0.55 \cdot e^{-0.65}]$	$[0.7 \cdot e^{0.6}, -0.6 \cdot e^{-0.5}]$

The Tables 6, 7 are the normalized decision values.

Table 6
Normalization of the Average Decision Values

$Option$	τ_1	τ_2
L	$(0.75,0.7), [0.7 \cdot e^{0.6125}, -0.5375 \cdot e^{-0.575}]$	$(0.65,0.55), [0.65,0.6875, -0.525 \cdot e^{-0.5625}]$
M	$(0.6,0.55), [0.5125 \cdot e^{0.5375}, -0.55 \cdot e^{-0.5875}]$	$(0.7,0.65), [0.575,0.5, -0.5375 \cdot e^{-0.5875}]$
N	$(0.8,0.75), [0.575 \cdot e^{0.55}, -0.5875 \cdot e^{-0.575}]$	$(0.8,0.6), [0.5125,0.575, -0.55 \cdot e^{-0.5875}]$
P	$(0.65,0.55), [0.5125 \cdot e^{0.5375}, -0.5625 \cdot e^{-0.5625}]$	$(0.85,0.8), [0.5125,0.575, -0.5 \cdot e^{-0.6}]$
Q	$(0.6,0.45), [0.575 \cdot e^{0.5875}, -0.55 \cdot e^{-0.6}]$	$(0.7,0.65)[0.5875,0.5625, -0.5 \cdot e^{-0.5375}]$

Table 7
Normalization of the Average Decision Values

Option	τ_3	τ_4
L	$(0.75,0.7), [0.625 \cdot e^{0.65}, -0.425 \cdot e^{-0.5625}]$	$(0.85,0.8), [0.6125 \cdot e^{0.6125}, -0.525 \cdot e^{-0.5}]$
M	$(0.6,0.55), [0.5875 \cdot e^{0.55}, -0.525 \cdot e^{-0.575}]$	$(0.7,0.65), [0.55 \cdot e^{0.5625}, -0.4875 \cdot e^{-0.5875}]$
N	$(0.55,0.45), [0.525 \cdot e^{0.55}, -0.5375 \cdot e^{-0.55}]$	$(0.65,0.6), [0.6375 \cdot e^{0.625}, -0.6125 \cdot e^{-0.55}]$
P	$(0.7,0.6), [0.45 \cdot e^{0.5125}, -0.55 \cdot e^{-0.575}]$	$(0.55,0.5), [0.6 \cdot e^{0.575}, -0.5375 \cdot e^{-0.55}]$
Q	$(0.8,0.75), [0.5125 \cdot e^{0.5375}, -0.55 \cdot e^{-0.5875}]$	$(0.65,0.6), [0.6375 \cdot e^{0.575}, -0.575 \cdot e^{-0.4625}]$

The normalised decision values are calculated and presented in Tables 8 and 9.

Table 8
Normalized Aggregate Decision Values

Option	τ_1	τ_2
L	$(0.9375,0.8711), [0.7 \cdot e^{0.6125}, -0.5375 \cdot e^{-0.575}]$	$(0.7647,0.5817), [0.65,0.6875, -0.525 \cdot e^{-0.5625}]$
M	$(0.7500,0.6722), [0.5125 \cdot e^{0.5375}, -0.55 \cdot e^{-0.5875}]$	$(0.8235,0.7545), [0.575,0.5, -0.5375 \cdot e^{-0.5875}]$
N	$(1.0000,0.9375), [0.575 \cdot e^{0.55}, -0.5875 \cdot e^{-0.575}]$	$(0.9412,0.5625), [0.5125,0.575, -0.55 \cdot e^{-0.5875}]$
P	$(0.8125,0.6205), [0.5125 \cdot e^{0.5375}, -0.5625 \cdot e^{-0.5625}]$	$(1.0000,0.9412), [0.5125,0.575, -0.5 \cdot e^{-0.6}]$
Q	$(0.7500,0.4500), [0.575 \cdot e^{0.5875}, -0.55 \cdot e^{-0.6}]$	$(0.8235,0.7545)[0.5875,0.5625, -0.5 \cdot e^{-0.5375}]$

Table 9
Normalized Aggregate Decision Values

Option	τ_3	τ_4
L	$(0.9375,0.8711), [0.625 \cdot e^{0.65}, -0.425 \cdot e^{-0.5625}]$	$(1.0000,0.9412), [0.6125 \cdot e^{0.6125}, -0.525 \cdot e^{-0.5}]$
M	$(0.7500,0.6722), [0.5875 \cdot e^{0.55}, -0.525 \cdot e^{-0.575}]$	$(0.8235,0.7545), [0.55 \cdot e^{0.5625}, -0.4875 \cdot e^{-0.5875}]$
N	$(0.6875,0.4909), [0.525 \cdot e^{0.55}, -0.5375 \cdot e^{-0.55}]$	$(0.7647,0.6923), [0.6375 \cdot e^{0.625}, -0.6125 \cdot e^{-0.55}]$
P	$(0.8750,0.6857), [0.45 \cdot e^{0.5125}, -0.55 \cdot e^{-0.575}]$	$(0.6471,0.5682), [0.6 \cdot e^{0.575}, -0.5375 \cdot e^{-0.55}]$
Q	$(1.0000,0.9375), [0.5125 \cdot e^{0.5375}, -0.55 \cdot e^{-0.5875}]$	$(0.7647,0.6923), [0.6375 \cdot e^{0.575}, -0.575 \cdot e^{-0.4625}]$

Use the RNBCFSWA operator to create data for each option in Table 10.

Table 10
RNBCFSWA Operator

L	$(0.9162,0.8233), [0.6354 \cdot e^{0.6438}, -0.4939 \cdot e^{-0.5410}]$
M	$(0.7941,0.7216), [0.5643 \cdot e^{0.5413}, -0.5170 \cdot e^{-0.5837}]$
N	$(0.8092,0.6240), [0.5699 \cdot e^{0.5838}, -0.5709 \cdot e^{-0.5616}]$
P	$(0.8202,0.7019), [0.5283 \cdot e^{0.5534}, -0.5339 \cdot e^{-0.5709}]$
Q	$(0.8485,0.7572), [0.5842 \cdot e^{0.5622}, -0.5455 \cdot e^{-0.5295}]$

$$\mathcal{S}_1^+ = 0.5226, \mathcal{S}_2^+ = 0.3977, \mathcal{S}_3^+ = 0.4095, \mathcal{S}_4^+ = 0.4048, \mathcal{S}_5^+ = 0.4411$$

and

$$\mathcal{S}_1^- = 0.4696, \mathcal{S}_2^- = 0.3614, \mathcal{S}_3^- = 0.3157, \mathcal{S}_4^- = 0.3465, \mathcal{S}_5^- = 0.3936.$$

The relative proximity values are

$$\mathcal{S}_1^* = 0.4961, \mathcal{S}_2^* = 0.3796, \mathcal{S}_3^* = 0.3626, \mathcal{S}_4^* = 0.3757, \mathcal{S}_5^* = 0.4174.$$

The choices are sorted as follows: $N < P < M < Q < L$.

5. IMPORTANCE OF THE WEIGHTED VECTOR

A notable limitation of DM methodologies that employ deterministic criteria to rank alternatives in practical decision environments is their potential to generate inconsistent outcomes, even when identical numerical information is considered. Consequently, appropriate validation requirements must be satisfied during the development and implementation of DM procedures to establish the reliability of their results. Table 11 presents the closeness values and corresponding alternative rankings obtained through the WA approach using the calculated score values. Since the selection outcomes depend on the assigned weight vectors, variations in these parameters can substantially influence the resulting preference order. To investigate this relationship, alternative rankings are determined using different weight configurations specified within the DM framework, with the corresponding selection outcomes reported in Table 11. These findings demonstrate the importance of assigning criterion weights that accurately reflect the practical conditions and priorities underlying the decision process.

Figures 2 and 3 further illustrate the changes in alternative rankings resulting from modifications to the weight vectors. The observed differences indicate that the proposed WA and WG operators exhibit considerable sensitivity to weight variations, suggesting a degree of instability in the resulting rankings. By systematically adjusting the weight vectors, the analysis demonstrates their influence on alternative preferences and identifies the range of possible ranking outcomes. Such findings emphasise the importance of accurately understanding the relative significance of individual evaluation criteria, as this knowledge enables decision-makers to establish more appropriate preferences and reach better-informed conclusions. Accordingly, sensitivity analysis is undertaken to assess the responsiveness of the alternative rankings to changes in the relative importance assigned to each criterion.

Table 11
Relative Closeness Values

Weighted Vectors	$\mathcal{S}_1^*$	$\mathcal{S}_2^*$	$\mathcal{S}_3^*$	$\mathcal{S}_4^*$	$\mathcal{S}_5^*$	Order
(0.1,0.35,0.3,0.25)	0.4807	0.3779	0.3606	0.3913	0.4189	N < M < P < Q < L
(0.1,0.35,0.25,0.3)	0.4816	0.3798	0.3653	0.3897	0.4156	N < M < P < Q < L
(0.1,0.25,0.35,0.3)	0.4952	0.3777	0.3579	0.3773	0.4206	N < P < M < Q < L
(0.1,0.25,0.3,0.35)	0.4961	0.3796	0.3626	0.3757	0.4174	N < P < M < Q < L
(0.1,0.3,0.35,0.25)	0.4875	0.3768	0.3569	0.3851	0.4213	N < M < P < Q < L
(0.1,0.3,0.25,0.35)	0.4893	0.3806	0.3664	0.3818	0.4148	N < M < P < Q < L
(0.25,0.35,0.3,0.1)	0.4757	0.3683	0.3738	0.3938	0.4032	M < N < P < Q < L
(0.25,0.3,0.35,0.1)	0.4825	0.3672	0.3702	0.3876	0.4056	M < N < P < Q < L
(0.25,0.1,0.3,0.35)	0.5141	0.3726	0.3793	0.355	0.3996	P < M < N < Q < L
(0.25,0.1,0.35,0.3)	0.5133	0.3707	0.3746	0.3568	0.4028	P < M < N < Q < L
(0.25,0.35,0.1,0.3)	0.4791	0.3759	0.393	0.3875	0.3899	M < P < Q < N < L
(0.25,0.3,0.1,0.35)	0.4868	0.3767	0.3942	0.3796	0.389	M < P < Q < N < L
(0.3,0.25,0.35,0.1)	0.4885	0.3649	0.3757	0.3808	0.3997	M < N < P < Q < L
(0.3,0.1,0.25,0.35)	0.5132	0.3713	0.3886	0.3542	0.3911	P < M < N < Q < L
(0.3,0.1,0.35,0.25)	0.5116	0.3675	0.3791	0.3577	0.3977	P < M < N < Q < L
(0.3,0.35,0.1,0.25)	0.4775	0.3727	0.3974	0.3885	0.3848	M < Q < P < N < L
(0.3,0.35,0.25,0.1)	0.4749	0.367	0.3829	0.3931	0.3948	M < N < P < Q < L
(0.3,0.25,0.1,0.35)	0.4927	0.3744	0.3997	0.3726	0.3831	P < M < Q < N < L
(0.35,0.3,0.25,0.1)	0.4809	0.3647	0.3884	0.3863	0.3889	M < P < N < Q < L
(0.35,0.25,0.3,0.1)	0.4878	0.3637	0.3848	0.3801	0.3914	M < P < N < Q < L

(0.35,0.1,0.3,0.25)	0.5108	0.3662	0.3883	0.357	0.3893	$P < M < N < Q < L$
(0.35,0.1,0.25,0.3)	0.5116	0.3681	0.3931	0.3552	0.386	$P < M < Q < N < L$
(0.35,0.3,0.1,0.25)	0.4834	0.3704	0.4029	0.3815	0.3789	$M < Q < P < N < L$
(0.35,0.25,0.1,0.3)	0.4911	0.3712	0.4041	0.3736	0.378	$M < P < Q < N < L$

Figures 2 and 3 illustrate the variations in the weighted vectors.

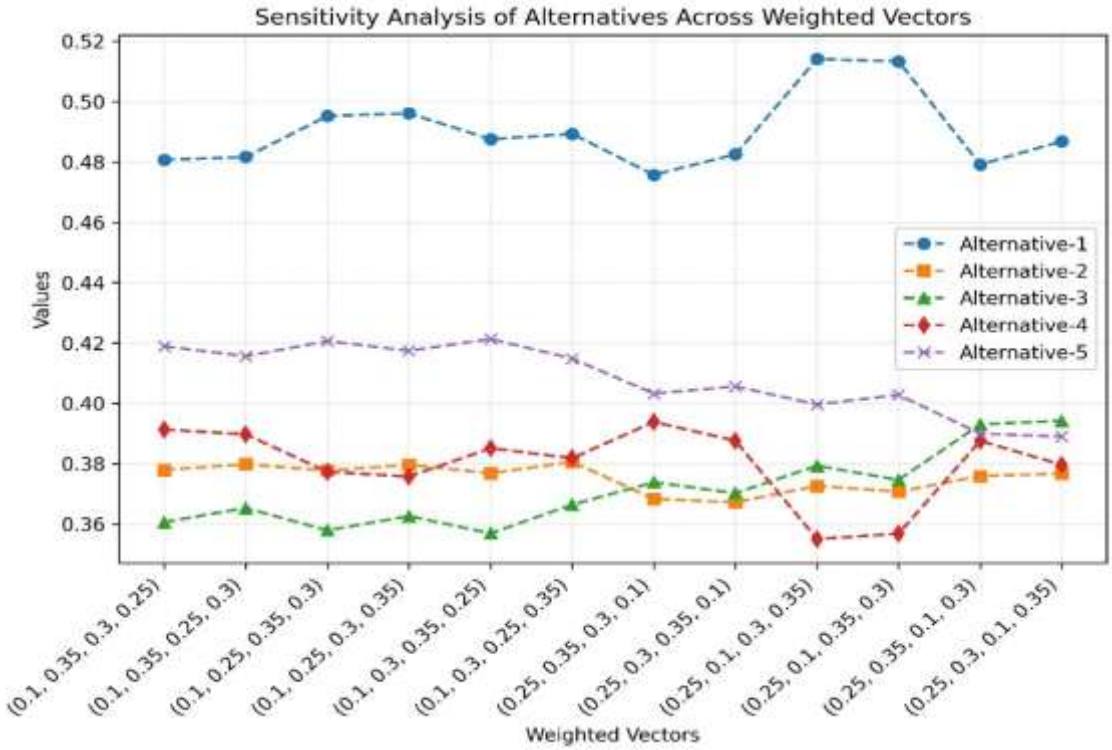


Fig.2: Different Weighted Vectors

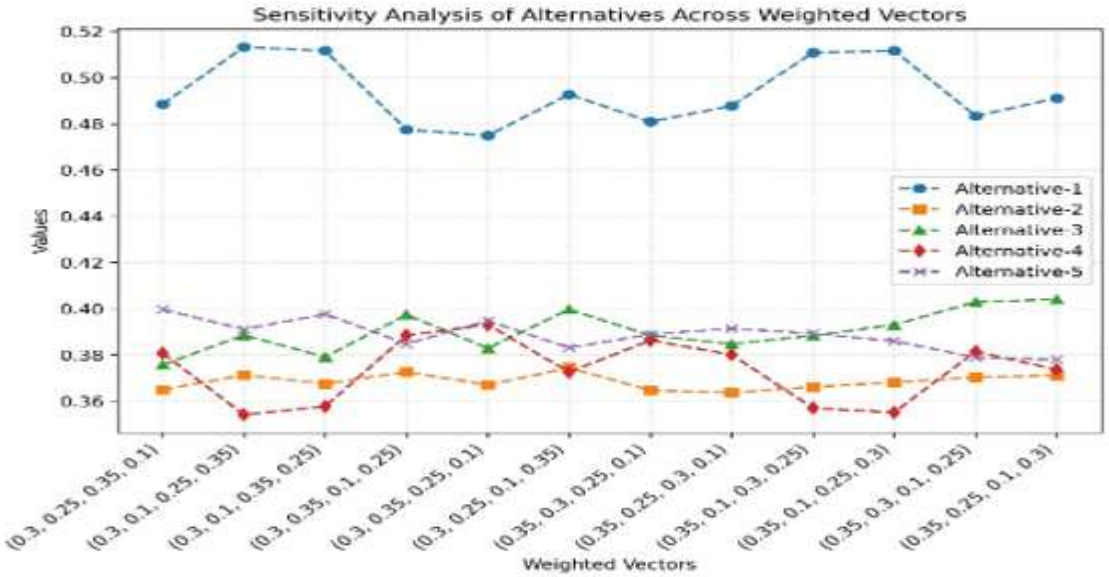


Fig.3: Different Weighted Vectors

5.1 Comparative Analysis

A comprehensive comparative investigation is conducted to assess the effectiveness and reliability of the proposed RNBCFSS framework and its associated AOs, namely RNBCFSWA and

RNBCFSWG, against several advanced decision-making methodologies. Particular emphasis is placed on evaluating the performance of the proposed WA-based formulation relative to established soft-set DM approaches, with the objective of determining its comparative effectiveness in processing ambiguous, uncertain, and parameterised information. Since soft-set decision environments frequently involve incomplete, conflicting, and imprecise assessments, methodological consistency and reliability are essential considerations when selecting an appropriate analytical framework. The proposed approach seeks to address these challenges by integrating multiple forms of information, including truth, indeterminacy, falsity, and conventional fuzzy membership representations, within a unified mathematical structure.

To improve the robustness and adaptability of the DM process, specialised aggregation mechanisms, including RNBCFSWA and RNBCFSWG, are developed to consolidate heterogeneous and uncertain evaluations. These operators facilitate the systematic integration of information obtained from multiple sources, thereby supporting decision-makers in identifying suitable alternatives under diverse conditions. Their effectiveness in managing uncertainty and ambiguity is subsequently examined through comparative distance-based evaluations. The resulting distance measures and corresponding ordering relationships are presented in Tables 12, 13, and 14.

Table 12
Different SV Techniques

Methods	Option-1	Option-2	Option-3	Option-4	Option-5
SV(WG)(Proposed)	0.4961	0.3796	0.3626	0.3757	0.4174
SV(WA)(Proposed)	0.4892	0.3785	0.3555	0.3673	0.4101
Alghazzawi et al. [3] SV(WA)	0.5707	0.5237	0.4995	0.4972	0.5194
Alghazzawi et al. [3] SV(WG)	0.5692	0.5230	0.4961	0.4933	0.5165
Yaqoob et al. [36] SV(WA)	0.3111	0.2512	0.2553	0.2442	0.2679
Yaqoob et al. [36] SV(WG)	0.3096	0.2506	0.2530	0.2418	0.2658
Köseoğlu et al. [13] SV(WA)	0.6396	0.5528	0.5768	0.5409	0.5732
Köseoğlu et al. [13] SV(WG)	0.6378	0.5517	0.5726	0.5363	0.5703

Table 13
Different AV Techniques

Methods	Option-1	Option-2	Option-3	Option-4	Option-5
AV(WG)(Proposed)	0.2946	0.2310	0.2341	0.2276	0.2480
AV(WA)(Proposed)	0.2908	0.2303	0.2292	0.2220	0.2439
Alghazzawi et al. [3] AV(WA)	0.5646	0.5407	0.5704	0.5311	0.5649
Alghazzawi et al. [3] AV(WG)	0.5647	0.5403	0.5676	0.5275	0.5625
Yaqoob et al. [36] AV(WA)	0.5785	0.5516	0.5716	0.5466	0.5554
Yaqoob et al. [36] AV(WA)	0.5782	0.5511	0.5696	0.5445	0.5545
Köseoğlu et al. [13] AV(WA)	0.4091	0.3057	0.3328	0.2927	0.3287
Köseoğlu et al. [13]	0.4068	0.3045	0.3280	0.2879	0.3253

Table 14
Different SV and AV Techniques

Methods	WA	WG
SV(Proposed)	$N < P < M < Q < L$	$N < P < M < Q < L$
Alghazzawi et al. [31] SV	$P < N < Q < M < L$	$P < N < Q < M < L$
Yaqoob et al. [32] SV	$P < M < N < Q < L$	$P < M < N < Q < L$
Koseoglu et al. [33] SV	$P < M < Q < N < L$	$P < M < Q < N < L$
AV(Proposed)	$P < M < N < Q < L$	$P < N < M < Q < L$
Alghazzawi et al. [31] AV	$P < M < L < Q < N$	$P < M < Q < L < N$
Yaqoob et al. [32] AV	$P < M < Q < N < L$	$P < M < Q < N < L$
Koseoglu et al. [33] AV	$P < M < N < Q < L$	$P < M < Q < N < L$

Stability of the Suggested Approach: The alternative rankings generated by the proposed methodology using different AOs are presented in Tables 12, 13, and 14. A principal observation is that the rankings derived from score and accuracy values maintain an identical preference order across all four AOs. This consistency indicates that the proposed conceptual framework produces reliable and reproducible ranking outcomes irrespective of the aggregation mechanism employed. Such behaviour demonstrates the robustness of the methodology, particularly when multiple evaluation attributes are considered under specified decision conditions. Conversely, the score-based rankings obtained using existing methodologies, as reported in Tables 12, 13, and 14, exhibit variations depending on the mathematical characteristics of the selected AO. These differences in alternative ordering suggest that the proposed framework offers greater ranking consistency and reliability than the conventional DM approaches considered in the comparative evaluation.

Capacity for Discrimination and Flexibility: The proposed methodology demonstrates considerable discriminatory capability when distinguishing among alternatives exhibiting intermediate performance levels. This characteristic is particularly important in multi-attribute group decision-making (MAGDM) environments, where competing alternatives may possess closely comparable attributes without exhibiting clear superiority or inferiority. The rankings established through accuracy functions exhibit a consistent relationship with those obtained from score functions, reinforcing the coherence of the evaluation process. Furthermore, the stability of accuracy-based assessments across the proposed AOs provides additional evidence of methodological reliability. Within the resulting preference structure, the most favourable alternative consistently occupies the highest position, whereas the least desirable option remains at the lowest rank, demonstrating the framework's ability to preserve relative alternative preferences.

The comparative results reported in Tables 12, 13, and 14 demonstrate that the proposed methodology maintains stable and consistent alternative rankings when evaluated using both score and accuracy functions. The comparative investigation establishes the stability and reliability of the proposed MCGDM framework through the consistent ranking outcomes obtained across the examined AOs. Its ability to preserve ranking precision and stability under uncertain conditions represents a significant advantage relative to conventional methodologies that exhibit variations in their evaluation outcomes. By combining consistent preference ordering with improved discrimination among competing alternatives, the proposed framework provides a flexible and analytically effective DM mechanism for complex environments involving multiple conflicting criteria. Figure 4 presents the SVs obtained from the existing and proposed models using the RNBCFSWA

operators.

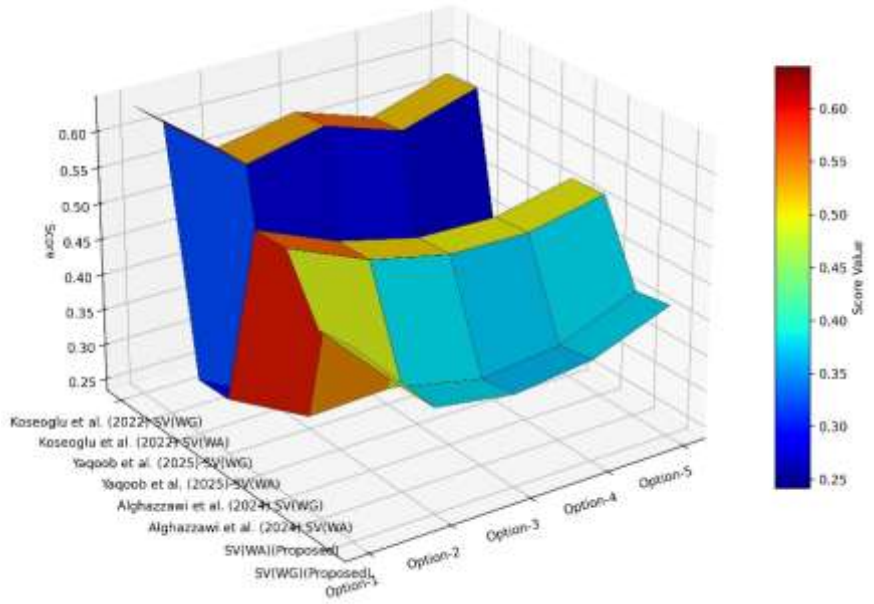


Fig.4: Different SVs

Figure 5 illustrates the AVs obtained using the RNBCFSWA operators for both the existing and proposed models.

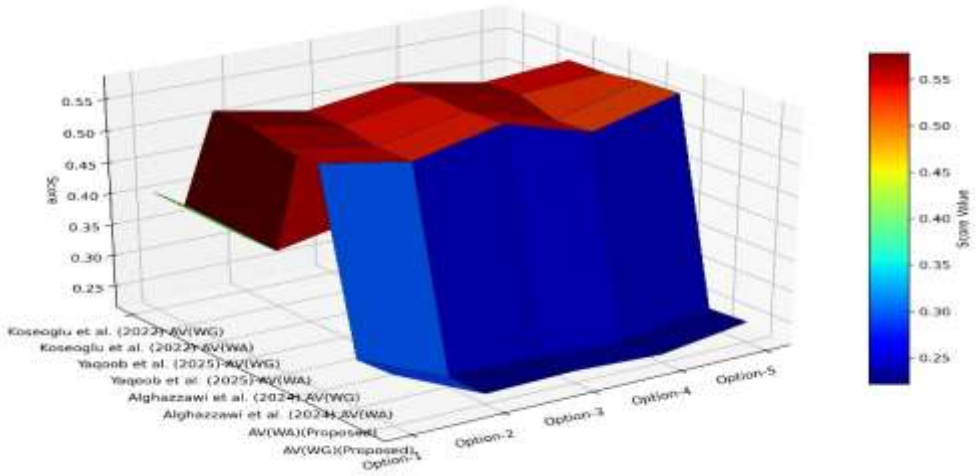


Fig.5: Different Avs

5.2 Sensitivity Analysis for Parameters and Experts

The applicability and robustness of the proposed methodologies are demonstrated through an algorithmic framework developed to address the MADM problem. By systematically varying the relevant parameters, decision-makers can obtain different score values and alternative rankings, thereby examining the responsiveness of the evaluation process to changes in decision conditions. A distinguishing characteristic of the proposed approach is its ability to maintain consistency and reliability throughout the assessment procedure. The RNBCFSWA and RNBCFSWG operators are

employed to aggregate the MADM information under different values of n, with the resulting score values and alternative rankings obtained using RNBCFSWA reported in Tables 12, 13, and 14. Examination of these outcomes indicates that the preferred alternative consistently outperforms the remaining options across the investigated parameter configurations.

Furthermore, the geometric interpretation of the parameter vector provides an important analytical perspective for understanding its influence on the decision-support framework. The effects of parameter variations on the overall evaluation function and final preference rankings are detailed in Tables 12, 13, and 14. The graphical results presented in Figures 4 and 5 further demonstrate the robustness of the proposed methodology by illustrating the stability of alternative rankings under variations in attribute and expert weights. To provide an additional quantitative assessment of ranking consistency, Spearman's rank correlation test [23] is employed to examine differences among the alternative preference orders generated by the various AOs within the developed framework. Spearman's rank correlation measures the strength and direction of the monotonic association between two ranked variables and is mathematically expressed as follows:

$$\rho = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)},$$

where n is the number of options and d is the ranking difference of an alternative between two different rankings. $\rho = +1$ indicates a perfect positive link between the two sets of rankings, whereas $\rho = -1$ [23] implies a perfect negative association. ρ is in the range of -1 to $+1$. When ρ is nearer ± 1, the two rankings are more correlated. The Spearman's rank correlation coefficient values between different ranks derived from different AOs are summarized in Tables 15 and 16. It is evident from these tables that the ranking orders generated using RNBCFSWA, RNBCFSWG, BFDWA, BFDWG, CBDFWAA, CBDFWGA, CFZNWA, and CFZNWG have a perfect positive association.

Table 15
Spearman's Rank Correlation Coefficient

$Parameter(\theta)$	RNBCFSWA	RNBCFSWG	BFDWA	BFDWG	CBDFWAA	CBDFWGA	CFZNWA	CFZNWG
RNBCFSWA	1	1	0	0	0.7	0.7	0.1	0.1
RNBCFSWG	1	1	0	0	0.7	0.7	0.1	0.1
BFDWA	0	0	1	1	0.3	0.3	0.1	0.1
BFDWG	0	0	1	1	0.3	0.3	0.1	0.1
CBDFWAA	0.7	0.7	0.3	0.3	1	1	0.1	0.1
CBDFWGA	0.7	0.7	0.3	0.3	1	1	0.1	0.1
CFZNWA	0.1	0.1	0.9	0.9	0.6	0.6	1	1
CFZNWG	0.1	0.1	0.9	0.9	0.6	0.6	1	1

Table 16
Spearman's Rank Correlation Coefficient

$Parameter(\theta)$	RNBCFSWA	RNBCFSWG	BFDWA	BFDWG	CBDFWAA	CBDFWGA	CFZNWA	CFZNWG
RNBCFSWA	1	0.9	0.6	-0.2	0.6	0.6	1	0.6
RNBCFSWG	0.9	1	0.7	-0.5	0.3	0.3	0.9	0.3
BFDWA	0.6	0.7	1	-0.6	-0.2	-0.2	0.6	-0.2
BFDWG	-0.2	-0.5	-0.6	1	0.6	0.6	-0.2	0.6
CBDFWAA	0.6	0.3	-0.2	0.6	1	1	0.6	1
CBDFWGA	0.6	0.3	-0.2	0.6	1	1	0.6	1
CFZNWA	1	0.9	0.6	-0.2	0.6	0.6	1	0.6
CFZNWG	0.6	0.3	-0.2	0.6	1	1	0.6	1

5.3 Needs, Limitations and Impact of the Work

The growing complexity of practical DM problems, particularly those encountered in investment strategy optimisation, finance, healthcare, and engineering, highlights the need for advanced mathematical frameworks capable of representing multiple dimensions of uncertainty simultaneously. Such problems frequently involve conflicting positive and negative evaluations, oscillatory or periodic characteristics, and ambiguous information that conventional approaches cannot always accommodate within a unified structure. Existing fuzzy, bipolar, and soft-set methodologies generally address these characteristics independently, potentially resulting in information loss, inconsistent alternative rankings, and limited interpretability.

The proposed RNBCFSS framework addresses these limitations by integrating bipolar membership assessments, soft-set parameterisation, and complex-valued amplitude–phase representations within a comprehensive mathematical formulation. This integration facilitates a more detailed and accurate representation of multidimensional decision environments involving uncertainty, duality, and periodicity. Nevertheless, several limitations remain associated with the proposed methodology. The computational complexity of its mathematical operations and aggregation procedures may constrain scalability when applied to substantially larger decision problems without further optimisation. Moreover, determining appropriate phase values requires reliable time-series observations or carefully established expert assessments, potentially necessitating additional calibration procedures and specialised domain knowledge. Despite these constraints, the proposed framework represents a meaningful methodological contribution. By providing a coherent, interpretable, and comparatively stable analytical structure, the CFBSS framework strengthens ranking reliability and supports more informed decisions under conditions characterised by uncertainty, bipolarity, and periodic variation.

5.4 Advantages

The proposed RNBCFSS framework provides several theoretical and practical advantages that strengthen the effectiveness and mathematical foundations of MCDM methodologies:

1. By incorporating fuzziness, bipolar membership information, parameterisation, and periodic characteristics within a unified mathematical representation, the framework minimises the potential loss of information associated with modelling these dimensions independently.
2. The establishment of explicitly defined operational laws, score and accuracy functions, entropy and distance measures, and ordering relations provides a coherent mathematical foundation that supports computational consistency and reliability throughout the evaluation process.
3. The newly formulated RNBCFSWA and RNBCFSWG operators enable the systematic aggregation of positive and negative membership assessments while preserving fundamental mathematical properties, including idempotency, boundedness, and monotonicity.
4. Comparative evaluations and sensitivity analyses indicate that the proposed methodology maintains consistent alternative rankings under the examined variations in expert preferences and parameter weights, thereby reducing ranking instability and strengthening decision reliability.
5. The complex-plane representation of membership information through amplitude and phase components enables intuitive graphical interpretation, improving the accessibility of evaluation outcomes and facilitating their communication to decision-makers.
6. The adaptability of the proposed framework supports its application across diverse sectors, including engineering, healthcare, renewable energy, and finance, particularly where decision problems involve simultaneous uncertainty, conflicting assessments, and oscillatory behaviour.

6. CONCLUSIONS AND FUTURE STUDIES

This investigation has developed and evaluated a novel mathematical framework for addressing MADM problems through a structured DM methodology, supported by computational experimentation and a practical case study. By integrating CFSs with BFSS, the proposed RNBCFSS formulation provides a unified mechanism for representing ambiguous information, bipolar evaluations, and periodic characteristics, thereby facilitating more consistent and interpretable decision outcomes. The methodological development encompasses a comprehensive collection of mathematical tools, including the RNBCFSWA and RNBCFSWG operators, together with explicitly formulated operational laws, score and accuracy functions, entropy measures, and distance metrics. The mathematical foundations of these aggregation mechanisms are established through the examination of essential algebraic properties, including commutativity, idempotency, boundedness, associativity, and monotonicity, which support their operational consistency and applicability. The effectiveness and robustness of the proposed framework and its associated AOs are subsequently investigated through parameter-dependent sensitivity analyses, enabling an assessment of their responsiveness to variations in decision conditions.

Building upon these theoretical developments, a systematic stepwise DM algorithm is constructed and incorporated into an MCGDM methodology to facilitate the evaluation and prioritisation of competing alternatives. Furthermore, the consideration of different weighted vector configurations demonstrates the flexibility of the proposed approach and its potential applicability across diverse decision environments. The comparative findings presented in Tables 12, 13, and 14 indicate that the proposed methodology achieves superior performance relative to the examined existing approaches, particularly in maintaining consistent alternative rankings and addressing complex MCGDM problems involving uncertainty. Future investigations will extend the developed framework to a wider range of practical applications, particularly within business management, manufacturing, and other decision-intensive sectors. Additional research will explore its integration with alternative fuzzy information environments, including hesitant fuzzy sets, hesitant picture fuzzy linguistic (HPFL) structures, picture fuzzy sets, and related mathematical formulations, with the objective of further strengthening the adaptability, computational robustness, and practical effectiveness of the proposed DM system.

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