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Educational Technology Assessment Using Bipolar Complex Pythagorean Neutrosophic Hamacher Aggregation Operators

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ABSTRACT

Multiple-Criteria Decision-Making (MCDM) problems frequently involve uncertainty, incomplete information and inconsistent evaluations, particularly when alternatives are assessed simultaneously from favourable and unfavourable perspectives. To address these complexities, this study develops an integrated decision-making framework based on Bipolar Complex Pythagorean Neutrosophic Sets (BCPNSs) and Hamacher Aggregation Operators (HAOs). The proposed framework incorporates bipolar evaluations, complex-valued membership degrees, Pythagorean characteristics and neutrosophic representations of truth, indeterminacy and falsity within a unified mathematical structure. Three aggregation mechanisms are formulated within this framework: the BCPNS Hamacher Weighted Averaging (BCPNHWA), BCPNS Hamacher Ordered Weighted Averaging (BCPNHOWA) and BCPNS Hamacher Hybrid Averaging (BCPNHHA) operators. Their fundamental mathematical properties, including idempotency, monotonicity and boundedness, are subsequently examined. The applicability of the proposed methodology is demonstrated through an educational technology selection problem comprising three candidate alternatives evaluated against three criteria. The BCPNHWA operator is employed to synthesise criterion-specific BCPNS assessments, after which a score function determines the relative preference ordering of the alternatives. The resulting scores are 0.3972, 0.4539 and 0.4569 for Ɒ_1, Ɒ_2 and Ɒ_3, respectively, establishing the ranking Ɒ_3 > Ɒ_2 > Ɒ_1. Accordingly, Ɒ_3, representing artificial intelligence (AI)-based adaptive assessment, emerges as the most preferable educational technology among the alternatives considered. By integrating complementary representations of uncertainty with flexible aggregation procedures, the proposed framework provides a mathematically versatile approach for managing ambiguity, conflicting assessments and imprecise information in educational decision-making and broader MCDM applications.

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1. Introduction

Decision-making problems frequently arise in environments characterised by uncertain, imprecise, ambiguous and inconsistent information, thereby restricting the effectiveness of conventional evaluation methodologies. Addressing these limitations requires analytical frameworks capable of accommodating multiple forms of uncertainty within a coherent mathematical structure. Accordingly, this study proposes an MCDM methodology integrating BCPNSs with HAOs. The framework simultaneously incorporates bipolar assessments, neutrosophic information and complex-valued membership functions to represent multidimensional uncertainty. Furthermore, a family of HAOs is formulated within the BCPNS environment to facilitate the systematic aggregation of evaluations across multiple decision criteria.

An important advancement in representing ambiguous information was achieved through the development of bipolar complex neutrosophic sets, which combine complex neutrosophic elements with their bipolar counterparts while incorporating amplitude and phase components [7]. Subsequent developments refined the mathematical representation of uncertain information through bipolar complex fuzzy sets (BCFSs), addressing limitations in existing approaches to modelling complex-valued and bipolar information [21]. The same investigation introduced similarity measures and demonstrated their practical relevance to disease identification and pattern recognition. Additional mathematical developments included geometric aggregation operators for LIVINFNs, together with scalar multiplication and neutral addition operations [3]. The integration of bipolar and complex fuzzy representations was further examined through the construction of BCFWAA and BCFOWGA operators for multi-attribute decision-making (MADM) problems [20].

In another direction, Dombi t-norm and t-conorm aggregation operations were formulated for trapezoidal-valued Fermatean fuzzy numbers, providing symmetrical mechanisms for synthesising ambiguous evaluations in multi-attribute group decision-making (MAGDM) environments [16]. A further extension employed complex p,q-rung orthopair fuzzy sets with Hamacher aggregation operations to accommodate uncertainty and complexity in expert judgements within MAGDM problems [25]. Induced bipolar neutrosophic Einstein aggregation operators were subsequently developed, with their effectiveness demonstrated through an illustrative multi-criteria group decision-making (MCGDM) application [13]. The literature also introduced several bipolar complex fuzzy HAOs, whose applicability was examined through a numerical example [19]. Complementary contributions established flexible t-norm and t-conorm operations alongside bipolar fuzzy Dombi aggregation operators [14]. Another significant theoretical development involved the introduction of Pythagorean neutrosophic sets, which integrate the principles of Pythagorean fuzzy sets with neutrosophic information [15].

The mathematical foundations underlying these developments originate from fuzzy set theory, which introduced membership degrees as a systematic means of representing imprecise information [27]. This foundation was subsequently extended through intuitionistic fuzzy sets, incorporating membership, non-membership and hesitation into uncertainty representation [5]. Neutrosophic set theory further broadened this perspective by independently characterising truth, indeterminacy and falsity [24]. Collectively, these theoretical advances established a basis for addressing decision-making problems involving incomplete, ambiguous and potentially inconsistent assessments. Related mathematical developments also included advanced relational operators within relational database management systems [8]. The integration of fuzzy and neutrosophic principles subsequently produced single-valued neutrosophic fuzzy sets, enabling uncertainty to be represented through truth, indeterminacy and falsity components defined over conventional unit intervals [9]. Another important generalisation introduced complex fuzzy sets, in which complex-valued membership

degrees capture uncertainty through amplitude and phase information [23]. This representation was extended to complex neutrosophic sets, where complex-valued membership functions provide greater flexibility for describing periodic, indeterminate and imprecise information [2].

The practical relevance of combining bipolar assessments with complex-valued membership information was further demonstrated through a sustainable supplier selection framework employing bipolar complex Pythagorean fuzzy sets [4]. Alongside these developments, hybrid averaging operators were introduced by integrating weighted averaging and ordered weighted averaging procedures [22]. This principle subsequently supported the formulation of FGHA, FIGHA, Quasi-FHA and Quasi-FIHA operators through fuzzy numbers, generalised and quasi-arithmetic means, and order-inducing variables. Neutrosophic uncertainty was also addressed through the construction of GNNHWA, GNNHOWA and GNNHHA operators based on Hamacher and generalised aggregation mechanisms [18]. Within bipolar fuzzy MADM environments, Hamacher aggregation was examined through weighting, ordering and hybrid aggregation procedures, with practical applicability demonstrated in enterprise resource planning (ERP) system selection [26].

Further research investigated aggregation mechanisms for complex fuzzy values using partial-order relationships and complex-valued membership functions, resulting in CFAA, CFWAA and CFOWAA operators [6]. That investigation additionally explored relationships between complex fuzzy and Pythagorean fuzzy representations and established the rotation invariance property. More recently, complex single-valued neutrosophic hesitant fuzzy numbers were developed to accommodate uncertainty within single-valued neutrosophic environments [12]. Hamacher sum and product operations were also established for fractional fuzzy sets, providing the mathematical basis for corresponding HAOs [1]. These aggregation mechanisms have supported decision-making applications involving extended versions of the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), TODIM and Grey Relational Analysis (GRA) approaches. Further developments incorporated hesitant bipolar complex fuzzy sets into AI-based energy management systems to accommodate hesitation in decision-making assessments [11]. A symmetrical TOPSIS methodology employing trapezoidal-valued neutrosophic fuzzy Aczel–Alsina aggregation operators was also proposed for intelligent recruitment decisions, demonstrating the applicability of Aczel–Alsina aggregation to information represented through truth, indeterminacy and falsity components [17].

Building upon these developments, the present investigation extends the existing methodological landscape by integrating BCPNS information with HAOs. Although considerable research has examined fuzzy, neutrosophic, bipolar, complex-valued and Hamacher-based aggregation frameworks, the available methodologies generally accommodate only selected combinations of these informational characteristics. Consequently, the present study introduces an integrated BCPNS-based aggregation framework incorporating BCPNHWA, BCPNHOWA and BCPNHHA operators. These operators are designed to synthesise bipolar, complex-valued and neutrosophic assessments under Pythagorean constraints. Their mathematical properties are investigated, and their application is demonstrated through an MCDM framework for educational technology evaluation.

1.1 Theoretical Foundation

The theoretical foundation of this investigation emerges from the progressive development of fuzzy, intuitionistic fuzzy, neutrosophic, complex-valued, bipolar and Pythagorean set theories, each contributing distinct mechanisms for representing uncertainty in decision-making environments. Their integration provides the conceptual basis for BCPNSs, which accommodate positive and negative evaluations, complex-valued membership information and independently specified truth,

indeterminacy and falsity components. Within this unified representation, the Hamacher framework provides the mathematical mechanism for aggregating uncertain evaluations across multiple decision criteria while retaining the underlying informational characteristics.

1.2 Novelty and Research Gap

Despite substantial advances in aggregation methodologies involving bipolar fuzzy, complex fuzzy, neutrosophic and Hamacher-based decision-making frameworks, existing approaches generally represent only subsets of the informational dimensions considered in this investigation. For example, bipolar fuzzy Hamacher aggregation operators accommodate favourable and unfavourable assessments but do not simultaneously incorporate complex-valued information, independent truth, indeterminacy and falsity components, and Pythagorean membership constraints [26].

Generalised neutrosophic Hamacher aggregation provides a comprehensive mechanism for processing truth, indeterminacy and falsity information; nevertheless, it does not explicitly integrate bipolar complex-valued assessments within a Pythagorean structure [18]. Similarly, complex fuzzy aggregation operators enable the manipulation of complex-valued membership information but do not jointly represent bipolar evaluations and neutrosophic components subject to Pythagorean restrictions [6]. Although bipolar complex fuzzy Hamacher aggregation combines bipolarity, complex-valued membership information and Hamacher operations, it does not explicitly accommodate independent neutrosophic truth, indeterminacy and falsity assessments alongside Pythagorean constraints [19]. Furthermore, hesitant bipolar complex fuzzy Hamacher power aggregation extends bipolar complex fuzzy information through hesitant evaluations and Hamacher power operations, but does not establish the specific BCPNS representation adopted in the present investigation [11].

The mathematical contribution of the proposed methodology therefore extends beyond a terminological combination of previously established fuzzy frameworks. Its distinguishing characteristic lies in the simultaneous representation of multiple informational dimensions within a unified mathematical model. Specifically, favourable and unfavourable assessments are incorporated through bipolar information structures, whereas complex-valued membership information is represented through the real and imaginary components of membership values. Independent truth, indeterminacy and falsity assessments are accommodated through the neutrosophic representation, while Pythagorean restrictions govern the admissibility of membership information. Consequently, the proposed HAOs operate within a BCPNS information environment that cannot be fully represented by the individual frameworks examined in the preceding studies.

To address this methodological limitation, the investigation develops three aggregation mechanisms, namely BCPNHWA, BCPNHOWA and BCPNHHA. Their mathematical properties are examined to establish the consistency and validity of the proposed aggregation procedures. The resulting operators are subsequently incorporated into an MCDM framework for evaluating educational technology alternatives. Accordingly, the principal contribution lies not merely in introducing additional aggregation operators within an established fuzzy environment, but in constructing an integrated Hamacher-based aggregation methodology capable of preserving and processing the combined informational characteristics of BCPNSs.

1.3 Motivation

- Educational technologies differ in their potential effects on learning outcomes, learner engagement and implementation costs. These variations necessitate a systematic analytical methodology for comparing available technologies and establishing their relative priorities.
- Educational technology assessments frequently involve uncertain, imprecise, hesitant and

contradictory judgements. Moreover, evaluators may express both favourable and unfavourable opinions concerning the same alternative, creating informational complexities that conventional decision-making approaches may not adequately accommodate.

- BCPNSs provide a unified mathematical representation capable of simultaneously incorporating bipolar assessments, complex-valued membership information, Pythagorean characteristics and independent neutrosophic truth, indeterminacy and falsity components.
- Hamacher aggregation mechanisms offer flexibility in synthesising multiple-criteria evaluations through alternative weighting and ordering procedures. This capability motivates the development of a dedicated Hamacher aggregation framework for information represented within the BCPNS environment.

1.4 Objectives of the Study

- To formulate Hamacher aggregation mechanisms for BCPNS information through the development of three operators, namely BCPNHWA, BCPNHOWA and BCPNHHA.
- To investigate the fundamental mathematical properties of the proposed operators, including idempotency, monotonicity and boundedness, while establishing their well-definedness and closure within the BCPNS framework.
- To develop an MCDM methodology capable of systematically evaluating, comparing and prioritising alternative educational technologies.
- To demonstrate the practical applicability of the proposed framework through an illustrative educational technology selection problem and examine the consistency of the resulting preference ordering using comparative and sensitivity analyses.

1.5 Significance of the Study

The significance of this investigation encompasses both theoretical advancement and practical decision-making applicability. From a theoretical perspective, the proposed methodology extends existing fuzzy and neutrosophic decision-making frameworks by integrating bipolar assessments, complex-valued information, Pythagorean membership restrictions and independently represented neutrosophic truth, indeterminacy and falsity components. The developed aggregation operators provide a systematic means of synthesising these multidimensional assessments while preserving the defining characteristics of the BCPNS information structure. This integration contributes to the mathematical development of aggregation methodologies intended for decision-making environments involving multiple interacting forms of uncertainty. From a practical perspective, the proposed framework offers a structured evaluation procedure for decision-makers who must compare alternatives using uncertain, ambiguous or conflicting expert judgements. In educational technology selection, candidate technologies can be assessed against multiple criteria, their evaluations aggregated and their relative preferences established through the resulting scores. Although the numerical illustration presented in this study does not employ an empirical educational dataset, it demonstrates how the proposed mathematical framework can support systematic decision-making when assessments contain conflicting or uncertain information.

1.6 Organization of the Paper

The remainder of this paper is organised as follows. Section 2 introduces the mathematical preliminaries associated with BCPNSs, score functions and Hamacher operations. Section 3 presents the proposed BCPNS HAOs and investigates their fundamental mathematical properties. Section 4 develops the corresponding MCDM methodology and demonstrates its application through an

educational technology selection problem. Finally, Section 5 summarises the principal conclusions of the investigation. To facilitate interpretation and maintain notational consistency, Table 1 provides a consolidated overview of the principal mathematical symbols and their corresponding meanings employed throughout the manuscript.

Table 1
Notation Table

Symbol	Meaning
$\mathfrak{S}$	Universal Set
$\mathbf{T}^{+}_{\Re(\wp)}(\mathfrak{e}) + \mathbf{i}\mathbf{T}^{+}_{\Im(\wp)}(\mathfrak{e})$	Complex-valued positive truth membership
$\mathbf{T}^{-}_{\Re(\wp)}(\mathfrak{e}) + \mathbf{i}\mathbf{T}^{-}_{\Im(\wp)}(\mathfrak{e})$	Complex-valued negative truth membership
$\mathbf{I}^{+}_{\Re(\wp)}(\mathfrak{e}) + \mathbf{i}\mathbf{I}^{+}_{\Im(\wp)}(\mathfrak{e})$	Complex-valued positive indeterminacy membership
$\mathbf{I}^{-}_{\Re(\wp)}(\mathfrak{e}) + \mathbf{i}\mathbf{I}^{-}_{\Im(\wp)}(\mathfrak{e})$	Complex-valued negative indeterminacy membership
$\mathbf{F}^{+}_{\Re(\wp)}(\mathfrak{e}) + \mathbf{i}\mathbf{F}^{+}_{\Im(\wp)}(\mathfrak{e})$	Complex-valued positive falsity membership
$\mathbf{F}^{-}_{\Re(\wp)}(\mathfrak{e}) + \mathbf{i}\mathbf{F}^{-}_{\Im(\wp)}(\mathfrak{e})$	Complex-valued negative falsity membership
$\eth_{\mathbf{s}}$	Hamacher parameter
$\boldsymbol{\xi}_{\mathfrak{b}}$	Weight vector
$\boldsymbol{\theta}_{\mathfrak{b}}$	Linked weight vector
$\mathfrak{Y}$	Alternative
$\mathbf{C}$	Criteria

2. PRELIMINARIES

Consider a discourse universe $\mathfrak{S}$, a fuzzy set $\wp$ in $\mathfrak{S}$ is discussed as a collection of arranged pairs: $\wp = \{\langle \mathfrak{e}, \mu_{\wp}(\mathfrak{e})\rangle \mid \mathfrak{e} \in \mathfrak{S}\}$, where $\mu_{\wp}(\mathfrak{e})$ denotes the membership grade of the element $\mathfrak{e}$ in $\wp$ such that $\mu_{\wp}: \mathfrak{S} \to [0,1]$, where the latter is the membership function. In this case, $\mu_{\wp}(\mathfrak{e})$ is the membership grade of $\mathfrak{e}$ in the fuzzy set $\wp$ [27].

Let $\mathfrak{S}$ be a discourse universe. A neutrosophic set $\wp$ in $\mathfrak{S}$ is distinguished by three independent functions: $\wp = \{\langle \mathfrak{e}, \mathrm{T}_{\wp}(\mathfrak{e}), \mathrm{I}_{\wp}(\mathfrak{e}), \mathrm{F}_{\wp}(\mathfrak{e})\rangle \mid \mathfrak{e} \in \mathfrak{S}\}$, where $\mathrm{T}_{\wp}(\mathfrak{e})$, $\mathrm{I}_{\wp}(\mathfrak{e})$, and $\mathrm{F}_{\wp}(\mathfrak{e})$ indicate the element's degree of T,I and F membership, $\mathfrak{e} \in \mathfrak{S}$, respectively.
These functions are all defined as:$\mathrm{T}_{\wp}(\mathfrak{e}): \mathfrak{S} \to [0,1], \mathrm{I}_{\wp}(\mathfrak{e}): \mathfrak{S} \to [0,1], \mathrm{F}_{\wp}(\mathfrak{e}): \mathfrak{S} \to [0,1]$, and satisfy the general condition: $0 \leq \mathrm{T}_{\wp}(\mathfrak{e}) + \mathrm{I}_{\wp}(\mathfrak{e}) + \mathrm{F}_{\wp}(\mathfrak{e}) \leq 3$ [24].

Let $\mathfrak{S}$ be a universal set. A complex neutrosophic set $\wp$ in $\mathfrak{S}$ is defined as: $\wp = \{\langle \mathfrak{e}, \mathrm{T}_{\Re(\wp)}(\mathfrak{e}) + \mathrm{i}\mathrm{T}_{\Im(\wp)}(\mathfrak{e}), \mathrm{I}_{\Re(\wp)}(\mathfrak{e}) + \mathrm{i}\mathrm{I}_{\Im(\wp)}(\mathfrak{e}), \mathrm{F}_{\Re(\wp)}(\mathfrak{e}) + \mathrm{i}\mathrm{F}_{\Im(\wp)}(\mathfrak{e})\rangle : \mathfrak{e} \in \mathfrak{S}\}$ where, $\mathrm{T}_{\Re(\wp)}(\mathfrak{e}), \mathrm{I}_{\Re(\wp)}(\mathfrak{e}), \mathrm{F}_{\Re(\wp)}(\mathfrak{e})$ are the real membership function and $\mathrm{T}_{\Im(\wp)}(\mathfrak{e}), \mathrm{I}_{\Im(\wp)}(\mathfrak{e}), \mathrm{F}_{\Im(\wp)}(\mathfrak{e})$ are the imaginary membership function [2].

Consider a discourse universe $\mathfrak{S}$, then the bipolar neutrosophic set is defined as $\wp = \{(\mathfrak{e}, (\mathrm{T}^{+}_{\wp}(\mathfrak{e}), \mathrm{T}^{-}_{\wp}(\mathfrak{e}); \mathrm{I}^{+}_{\wp}(\mathfrak{e}), \mathrm{I}^{-}_{\wp}(\mathfrak{e}); \mathrm{F}^{+}_{\wp}(\mathfrak{e}), \mathrm{F}^{-}_{\wp}(\mathfrak{e}))) \mid \mathfrak{e} \in \mathfrak{S}\}$ $\mathrm{T}^{+}_{\wp}(\mathfrak{e}), \mathrm{I}^{+}_{\wp}(\mathfrak{e}), \mathrm{F}^{+}_{\wp}(\mathfrak{e}): \mathfrak{S} \to [0,1]$ represent the functions of positive truth-membership, falsity-membership, and indeterminacy-membership, respectively. It stands for the support of a set element.

$\mathrm{T}^{-}_{\wp}(\mathfrak{e}), \mathrm{I}^{-}_{\wp}(\mathfrak{e}), \mathrm{F}^{-}_{\wp}(\mathfrak{e}): \mathfrak{S} \to [-1,0]$ represent the functions of falsity-membership, indeterminacy-membership, and negative truth-membership, respectively. It stands for the opposition or opposing viewpoint [10].
The structure of a bipolar complex neutrosophic set is represented as follows:

$$\wp = \left(\begin{array}{c} \mathfrak{e}, (\{\mathrm{T}^{+}_{\Re(\wp)}(\mathfrak{e}) + \mathrm{i}\mathrm{T}^{+}_{\Im(\wp)}(\mathfrak{e}), \mathrm{T}^{-}_{\Re(\wp)}(\mathfrak{e}) + \mathrm{i}\mathrm{T}^{-}_{\Im(\wp)}(\mathfrak{e})\}; \\ \{\mathrm{I}^{+}_{\Re(\wp)}(\mathfrak{e}) + \mathrm{i}\mathrm{I}^{+}_{\Im(\wp)}(\mathfrak{e}), \mathrm{I}^{-}_{\Re(\wp)}(\mathfrak{e}) + \mathrm{i}\mathrm{I}^{-}_{\Im(\wp)}(\mathfrak{e})\}; \\ \{\mathrm{F}^{+}_{\Re(\wp)}(\mathfrak{e}) + \mathrm{i}\mathrm{F}^{+}_{\Im(\wp)}(\mathfrak{e}), \mathrm{F}^{-}_{\Re(\wp)}(\mathfrak{e}) + \mathrm{i}\mathrm{F}^{-}_{\Im(\wp)}(\mathfrak{e})\}) \mid \mathfrak{e} \in \mathfrak{S} \end{array} \right)$$

where $\left(\mathrm{T}^{+}_{\Re(\wp)}(\ə), \mathrm{I}^{+}_{\Re(\wp)}(\ə), \mathrm{F}^{+}_{\Re(\wp)}(\ə)\right)$ are the real positive membership of BCPNS and $\left(\mathrm{T}^{+}_{\Im(\wp)}(\ə), \mathrm{I}^{+}_{\Im(\wp)}(\ə), \mathrm{F}^{+}_{\Im(\wp)}(\ə)\right)$ are the imaginary positive membership of BCPNS. Similarly, for negative part there exists real and imaginary membership function [7].

Let $\mathfrak{S}$ be a discourse universe. A Pythagorean neutrosophic set $\wp$ in $\mathfrak{S}$ is characterized by three independent functions:

$$\wp = \left\{\left\langle \ə, \mathrm{T}_{\wp}(\ə), \mathrm{I}_{\wp}(\ə), \mathrm{F}_{\wp}(\ə)\right\rangle \mid \ə \quad \in \mathfrak{S}\right\},$$

where $\mathrm{T}_{\wp}(\ə)$, $\mathrm{I}_{\wp}(\ə)$, and $\mathrm{F}_{\wp}(\ə)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees of the element $\ə \quad \in \mathfrak{S}$, respectively. These functions are defined as $\mathrm{T}_{\wp}(\ə): \mathfrak{S} \to [0,1], \mathrm{I}_{\wp}(\ə): \mathfrak{S} \to [0,1], \mathrm{F}_{\wp}(\ə): \mathfrak{S} \to [0,1]$, and satisfy the Pythagorean condition

$$0 \leq \mathrm{T}^{2}_{\wp}(\ə) + \mathrm{I}^{2}_{\wp}(\ə) + \mathrm{F}^{2}_{\wp}(\ə) \leq 3 \text{ [7].}$$

Let $\wp_{\mathrm{n}} \in [0,1], \mathrm{n} = 1,2$ pair of non-integers of any kind. Afterward, $\mathbb{TN}$ and $\mathbb{TCN}$ for $\eth_{\mathrm{s}} > 0$ are defined as:

$$\mathbb{H}_{\mathbb{TN}}(\wp_1, \wp_2) = \frac{\wp_1 * \wp_2}{\eth_{\mathrm{s}} + (1 - \eth_{\mathrm{s}})(\wp_1 + \wp_2 - \wp_1 * \wp_2)};$$
$$\mathbb{H}_{\mathbb{TCN}}(\wp_1, \wp_2) = \frac{\wp_1 + \wp_2 - \wp_1 * \wp_2 - (1 - \eth_{\mathrm{s}}) * \wp_1 * \wp_2}{1 - (1 - \eth_{\mathrm{s}})(\wp_1 * \wp_2)}$$

The Einstein and algebraic norms are therefore easily determined for $\eth_{\mathrm{s}} = 1,2$ [8].

3. HAMACHER AGGREGATION OPERATORS FOR BIPOLAR COMPLEX PYTHAGOREAN NEUTROSOPHIC SETS

HAOs provide a systematic mathematical mechanism for synthesising BCPNS information while preserving its underlying structural characteristics. By employing Hamacher t-norm and t-conorm operations, these operators facilitate the aggregation of positive and negative evaluations associated with truth, indeterminacy and falsity components under Pythagorean constraints. This integrated formulation enables the representation and processing of uncertainty and inconsistency inherent in decision-making assessments without compromising the defining properties of BCPNSs. Accordingly, this section introduces three Hamacher-based aggregation mechanisms within the BCPNS framework, namely BCPNHWA, BCPNHOWA and BCPNHHA, to accommodate different weighting and ordering requirements in the aggregation of multidimensional uncertain information.

3.1 Definition

Consider a group of BCPNHNs

$$\wp_{\mathfrak{b}} = \begin{pmatrix} \left\{\mathrm{T}^{+}_{\Re(\wp_{\mathfrak{b}})} + \mathrm{i}\mathrm{T}^{+}_{\Im(\wp_{\mathfrak{b}})}, \mathrm{T}^{-}_{\Re(\wp_{\mathfrak{b}})} + \mathrm{i}\mathrm{T}^{-}_{\Im(\wp_{\mathfrak{b}})}\right\}, \\ \left\{\mathrm{I}^{+}_{\Re(\wp_{\mathfrak{b}}))} + \mathrm{i}\mathrm{I}^{+}_{\Im(\wp_{\mathfrak{b}})}, \mathrm{I}^{-}_{\Re(\wp_{\mathfrak{b}})} + \mathrm{i}\mathrm{I}^{-}_{\Im(\wp_{\mathfrak{b}})}\right\}, \\ \left\{\mathrm{F}^{+}_{\Re(\wp_{\mathfrak{b}})} + \mathrm{i}\mathrm{F}^{+}_{\Im(\wp_{\mathfrak{b}})}, \mathrm{F}^{-}_{\Re(\wp_{\mathfrak{b}})} + \mathrm{i}\mathrm{F}^{-}_{\Im(\wp_{\mathfrak{b}})}\right\} \end{pmatrix}, \mathfrak{b} = 1,2,\cdots,\hat{\mathfrak{b}}$$

then

$$\mathrm{BCPNHWA}(\wp_1, \wp_2, \cdots, \wp_{\hat{\mathfrak{b}}}) = \bigoplus_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}} \xi_{\mathfrak{b}} \wp_{\mathfrak{b}} \qquad (1)$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_{\hat{\mathfrak{b}}})$ is a weight vector, $\xi_{\hat{\mathfrak{b}}} \in [0,1]$ and $\sum_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}} \xi_{\mathfrak{b}} = 1$

3.2 Definition

The score values associated with a BCPNS are mathematically defined as follows:

$$S(\wp)=\left[\begin{array}{l}\frac{1}{4}\left(2+(T^{+}_{\Re(\wp)})^{2}+(T^{+}_{\Im(\wp)})^{2}+(T^{-}_{\Re(\wp}})^{2}+(T^{-}_{\Im(\wp)})^{2}\right),\\ \frac{1}{4}\left(2+(I^{+}_{\Re(\wp)})^{2}+(I^{+}_{\Im(\wp)})^{2}+(I^{-}_{\Re(\wp}})^{2}+(I^{-}_{\Im(\wp)})^{2}\right),\\ \frac{1}{4}\left(2+(F^{+}_{\Re(\wp)})^{2}+(F^{+}_{\Im(\wp)})^{2}+(F^{-}_{\Re(\wp}})^{2}+(F^{-}_{\Im(\wp)})^{2}\right)\end{array}\right],\in[0,1]$$

3.3 Theorem

Consider a gathering of BCPNHNs

$$\wp_{\mathfrak{b}}=\begin{pmatrix}\left\{T^{+}_{\Re(\wp_{\mathfrak{b}})}+iT^{+}_{\Im(\wp_{\mathfrak{b}})},T^{-}_{\Re(\wp_{\mathfrak{b}})}+iT^{-}_{\Im(\wp_{\mathfrak{b}})}\right\},\\ \left\{I^{+}_{\Re(\wp_{\mathfrak{b}})}+iI^{+}_{\Im(\wp_{\mathfrak{b}})},I^{-}_{\Re(\wp_{\mathfrak{b}})}+iI^{-}_{\Im(\wp_{\mathfrak{b}})}\right\},\\ \left\{F^{+}_{\Re(\wp_{\mathfrak{b}})}+iF^{+}_{\Im(\wp_{\mathfrak{b}})},F^{-}_{\Re(\wp_{\mathfrak{b}})}+iF^{-}_{\Im(\wp_{\mathfrak{b}})}\right\}\end{pmatrix},\mathfrak{b}=1,2,\cdots,\hat{\mathfrak{b}}$$

then aggregating this with then help of equation (1), we get

$$\mathrm{BCPNHWA}(\wp_1,\wp_2,\cdots,\wp_{\hat{\mathfrak{b}}})=$$

$$\left(\begin{array}{l}\left(\frac{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(T^{+}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)-\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(T^{+}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(T^{+}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(T^{+}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}+i\left(\frac{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(T^{+}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)-\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(T^{+}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(T^{+}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(T^{+}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}},\\ \left(-\frac{(\delta_s)^{\frac{1}{2}}\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|T^{-}_{\Re(\wp_{\mathfrak{b}})}|^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(1+(T^{-}_{\Re(\wp_{\mathfrak{b}})})^2))^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|(T^{-}_{\Re(\wp_{\mathfrak{b}})})^2|^{\xi_{\mathfrak{b}}}\right)}\right)+i\left(-\frac{(\delta_s)^{\frac{1}{2}}\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|T^{-}_{\Im(\wp_{\mathfrak{b}})}|^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(1+(T^{-}_{\Im(\wp_{\mathfrak{b}})})^2))^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|(T^{-}_{\Im(\wp_{\mathfrak{b}})})^2|^{\xi_{\mathfrak{b}}}\right)}\right);\\ \left(\frac{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(I^{+}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)-\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(I^{+}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(I^{+}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(I^{+}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}+i\left(\frac{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(I^{+}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)-\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(I^{+}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(I^{+}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(I^{+}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}},\\ \left(-\frac{(\delta_s)^{\frac{1}{2}}\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|I^{-}_{\Re(\wp_{\mathfrak{b}})}|^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(1+(I^{-}_{\Re(\wp_{\mathfrak{b}})})^2))^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|(I^{-}_{\Re(\wp_{\mathfrak{b}})})^2|^{\xi_{\mathfrak{b}}}\right)}\right)+i\left(-\frac{(\delta_s)^{\frac{1}{2}}\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|I^{-}_{\Im(\wp_{\mathfrak{b}})}|^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(1+(I^{-}_{\Im(\wp_{\mathfrak{b}})})^2))^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|(I^{-}_{\Im(\wp_{\mathfrak{b}})})^2|^{\xi_{\mathfrak{b}}}\right)}\right);\\ \left(\frac{(\delta_s)^{\frac{1}{2}}\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(F^{+}_{\Re(\wp_{\mathfrak{b}})})^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(1+F^{+}_{\Re(\wp_{\mathfrak{b}})}))^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(F^{+}_{\Re(\wp_{\mathfrak{b}})})^{\xi_{\mathfrak{b}}}\right)}\right)+i\left(\frac{(\delta_s)^{\frac{1}{2}}\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(F^{+}_{\Im(\wp_{\mathfrak{b}})})^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)(1+F^{+}_{\Im(\wp_{\mathfrak{b}})}))^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(F^{+}_{\Im(\wp_{\mathfrak{b}})})^{\xi_{\mathfrak{b}}}\right)}\right),\\ \left(-\frac{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)|(F^{-}_{\Re(\wp_{\mathfrak{b}})})^2|)^{\xi_{\mathfrak{b}}}\right)-\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(F^{-}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)|(F^{-}_{\Re(\wp_{\mathfrak{b}})})^2|)^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(F^{-}_{\Re(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}+i\left(-\frac{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)|(F^{-}_{\Im(\wp_{\mathfrak{b}})})^2|)^{\xi_{\mathfrak{b}}}\right)-\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(F^{-}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\delta_s-1)|(F^{-}_{\Im(\wp_{\mathfrak{b}})})^2|)^{\xi_{\mathfrak{b}}}\right)+(\delta_s-1)\left(\prod_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(F^{-}_{\Im(\wp_{\mathfrak{b}})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}\end{array}\right)$$

Proof. The BCPNHWA operator holds three properties,

Proof. The BCPNHWA operator holds three properties,

3.3.1 Consider a Gathering of BCPNHNs

$$\wp_{\flat}=\begin{pmatrix}\{T^{+}_{\Re(\wp_{\flat})}+iT^{+}_{\Im(\wp_{\flat})},T^{-}_{\Re(\wp_{\flat})}+iT^{-}_{\Im(\wp_{\flat})}\},\\ \{I^{+}_{\Re(\wp_{\flat})}+iI^{+}_{\Im(\wp_{\flat})},I^{-}_{\Re(\wp_{\flat})}+iI^{-}_{\Im(\wp_{\flat})}\},\\ \{F^{+}_{\Re(\wp_{\flat})}+iF^{+}_{\Im(\wp_{\flat})},F^{-}_{\Re(\wp_{\flat})}+iF^{-}_{\Im(\wp_{\flat})}\}\end{pmatrix},\flat=1,2,\cdots,\hat{\flat}$$

if $\forall\ \flat,\wp_{\flat}=\wp$, then $\mathrm{BCPNHWA}(\wp_1,\wp_2,\cdots,\wp_{\hat{\flat}})=\wp$
The above property is known as idempotency.

3.3.2 Consider Two Groups of BCPNHNs

$$\wp_{\flat}=\begin{pmatrix}\{T^{+}_{\Re(\wp_{\flat})}+iT^{+}_{\Im(\wp_{\flat})},T^{-}_{\Re(\wp_{\flat})}+iT^{-}_{\Im(\wp_{\flat})}\},\\ \{I^{+}_{\Re(\wp_{\flat})}+iI^{+}_{\Im(\wp_{\flat})},I^{-}_{\Re(\wp_{\flat})}+iI^{-}_{\Im(\wp_{\flat})}\},\\ \{F^{+}_{\Re(\wp_{\flat})}+iF^{+}_{\Im(\wp_{\flat})},F^{-}_{\Re(\wp_{\flat})}+iF^{-}_{\Im(\wp_{\flat})}\}\end{pmatrix}$$

$$\text{and}\quad \wp^{*}_{\flat}=\begin{pmatrix}\{T^{+}_{\Re(\wp^{*}_{\flat})}+iT^{+}_{\Im(\wp^{*}_{\flat})},T^{-}_{\Re(\wp^{*}_{\flat})}+iT^{-}_{\Im(\wp^{*}_{\flat})}\},\\ \{I^{+}_{\Re(\wp^{*}_{\flat})}+iI^{+}_{\Im(\wp^{*}_{\flat})},I^{-}_{\Re(\wp^{*}_{\flat})}+iI^{-}_{\Im(\wp^{*}_{\flat})}\},\\ \{F^{+}_{\Re(\wp^{*}_{\flat})}+iF^{+}_{\Im(\wp^{*}_{\flat})},F^{-}_{\Re(\wp^{*}_{\flat})}+iF^{-}_{\Im(\wp^{*}_{\flat})}\}\end{pmatrix},\flat=1,2,\cdots,\hat{\flat}$$

$$\text{if }\forall\ \flat,T^{+}_{\Re(\wp_{\flat})}\le T^{+}_{\Re(\wp^{*}_{\flat})},T^{+}_{\Im(\wp_{\flat})}\le T^{+}_{\Im(\wp^{*}_{\flat})},T^{-}_{\Re(\wp_{\flat})}\le T^{-}_{\Re(\wp_{\flat})},T^{-}_{\Im(\wp_{\flat})}\le T^{-}_{\Im(\wp^{*}_{\flat})};\forall\ \flat,I^{+}_{\Re(\wp_{\flat})}\le I^{+}_{\Re(\wp^{*}_{\flat})},I^{+}_{\Im(\wp_{\flat})}\le I^{+}_{\Im(\wp^{*}_{\flat})},I^{-}_{\Re(\wp_{\flat})}\le I^{-}_{\Re(\wp_{\flat})},I^{-}_{\Im(\wp_{\flat})}\le I^{-}_{\Im(\wp^{*}_{\flat})}\text{ and }\forall\ \flat,F^{+}_{\Re(\wp_{\flat})}\le F^{+}_{\Re(\wp^{*}_{\flat})},F^{+}_{\Im(\wp_{\flat})}\le F^{+}_{\Im(\wp^{*}_{\flat})},F^{-}_{\Re(\wp_{\flat})}\le F^{-}_{\Re(\wp_{\flat})},F^{-}_{\Im(\wp_{\flat})}\le F^{-}_{\Im(\wp^{*}_{\flat})}\text{ then}$$

$$\mathrm{BCPNHWA}(\wp_1,\wp_2,\cdots,\wp_{\hat{\flat}})\le\mathrm{BCPNHWA}(\mathfrak{B}^{*}_1,\mathfrak{B}^{*}_2,\cdots,\mathfrak{B}^{*}_{\hat{\flat}})$$

This property is called monotonicity.

3.3.3 Consider a Gathering of BCPNHNs

$$\wp_{\flat}=\begin{pmatrix}\{T^{+}_{\Re(\wp_{\flat})}+iT^{+}_{\Im(\wp_{\flat})},T^{-}_{\Re(\wp_{\flat})}+iT^{-}_{\Im(\wp_{\flat})}\},\\ \{I^{+}_{\Re(\wp_{\flat})}+iI^{+}_{\Im(\wp_{\flat})},I^{-}_{\Re(\wp_{\flat})}+iI^{-}_{\Im(\wp_{\flat})}\},\\ \{F^{+}_{\Re(\wp_{\flat})}+iF^{+}_{\Im(\wp_{\flat})},F^{-}_{\Re(\wp_{\flat})}+iF^{-}_{\Im(\wp_{\flat})}\}\end{pmatrix},\flat=1,2,\cdots,\hat{\flat}$$

then assume

$$\wp^{-}=\begin{pmatrix}\begin{pmatrix}\min\limits_{\flat}\{T^{+}_{\Re(\wp_{\flat})}\}+i\min\limits_{\flat}\{T^{+}_{\Im(\wp_{\flat})}\},\\ \max\limits_{\flat}\{T^{-}_{\Re(\wp_{\flat})}\}+i\max\limits_{\flat}\{T^{-}_{\Im(\wp_{\flat})}\}\end{pmatrix},\\ \begin{pmatrix}\min\limits_{\flat}\{I^{+}_{\Re(\wp_{\flat})}\}+i\min\limits_{\flat}\{I^{+}_{\Im(\wp_{\flat})}\},\\ \max\limits_{\flat}\{I^{-}_{\Re(\wp_{\flat})}\}+i\max\limits_{\flat}\{I^{-}_{\Im(\wp_{\flat})}\}\end{pmatrix},\\ \begin{pmatrix}\min\limits_{\flat}\{F^{+}_{\Re(\wp_{\flat})}\}+i\min\limits_{\flat}\{F^{+}_{\Im(\wp_{\flat})}\},\\ \max\limits_{\flat}\{F^{-}_{\Re(\wp_{\flat})}\}+i\max\limits_{\flat}\{F^{-}_{\Im(\wp_{\flat})}\}\end{pmatrix}\end{pmatrix}$$

$$\wp^{+}=\left(\begin{array}{c}\left(\begin{array}{c}\max\limits_{\flat}\{\mathrm{T}^{+}_{\Re(\wp_{\flat})}\}+\max\limits_{\flat}\{\mathrm{T}^{+}_{\Im(\wp_{\flat})}\},\\ \min\limits_{\flat}\{\mathrm{T}^{-}_{\Re(\wp_{\flat})}\}+\min\limits_{\flat}\{\mathrm{T}^{-}_{\Im(\wp_{\flat})}\}\end{array}\right),\\ \left(\begin{array}{c}\max\limits_{\flat}\{\mathrm{I}^{+}_{\Re(\wp_{\flat})}\}+\max\limits_{\flat}\{\mathrm{I}^{+}_{\Im(\wp_{\flat})}\},\\ \min\limits_{\flat}\{\mathrm{I}^{-}_{\Re(\wp_{\flat})}\}+\min\limits_{\flat}\{\mathrm{I}^{-}_{\Im(\wp_{\flat})}\}\end{array}\right),\\ \left(\begin{array}{c}\max\limits_{\flat}\{\mathrm{F}^{+}_{\Re(\wp_{\flat})}\}+\max\limits_{\flat}\{\mathrm{F}^{+}_{\Im(\wp_{\flat})}\},\\ \min\limits_{\flat}\{\mathrm{F}^{-}_{\Re(\wp_{\flat})}\}+\min\limits_{\flat}\{\mathrm{F}^{-}_{\Im(\wp_{\flat})}\}\end{array}\right)\end{array}\right)$$

then $\wp^{-}\leq \mathrm{BCPNHWA}(\wp_1,\wp_2,\cdots,\wp_{\hat{\mathrm{b}}})\leq \wp^{+}$
This property is called Boundedness.

3.4 Verification of Well-Definedness and Closure of the Proposed Operators

Before establishing the algebraic properties of the proposed aggregation operators, it is necessary to verify that the aggregation process preserves the structural requirements of BCPNNs. Let $\wp_{\flat},\flat=1,2,\cdots,\hat{\mathrm{b}}$ be BCPNNs and let $\xi=(\xi_1,\xi_2,\ldots,\xi_b)$ be the associated weight vector satisfying $\xi_{\mathrm{l}}\in[0,1]$ and $\sum\xi_{\mathrm{l}}=1$. The Hamacher parameter is considered as $\eth_{\mathrm{s}}>0$, as defined in the Hamacher operations. Since the Hamacher t-norm and t-conorm preserve the unit interval, the component-wise aggregation used in the proposed BCPNHWA operator preserves the admissible domains of all membership components.

3.4.1 Proposition

The BCPNHWA operator defined by Equation (2) is well-defined and its aggregated value is a valid BCPNN.

Proof. For every input BCPNN $\wp_{\flat}$, the real and imaginary components of the positive truth, indeterminacy, and falsity memberships belong to $[0,1]$, whereas the corresponding negative components belong to $[-1,0]$. The BCPNHWA operator is constructed by applying Hamacher operations to these components and subsequently combining the weighted values. For $\eth_{\mathrm{s}}>0$ and $\xi_{\mathrm{l}}\in[0,1]$, the Hamacher t-norm and t-conorm map $[0,1]^2$ into $[0,1]$. Therefore, each positive real and positive imaginary component generated by Equation (2) remains in $[0,1]$. As far as the negative part is concerned, the aggregation formulas for the negative part will be created by taking the absolute value of the respective negative membership values and then the negative sign will be put to those values. As the aggregate absolute values are still within $[0,1]$, the aggregate negative real and negative imaginary parts will be within $[-1,0]$.

Moreover, the Pythagorean neutrosophic components of a BCPNN satisfy $0\leq \mathrm{T}_{\wp}(\ə)+\mathrm{I}_{\wp}(\ə)+\mathrm{F}_{\wp}(\ə)\leq 3$ Because the aggregated truth, indeterminacy, and falsity components generated by the Hamacher operations remain in [0,1] for the positive part and in [−1,0] for the negative part, their squared values remain in [0,1]. Consequently, the aggregated components satisfy the same Pythagorean neutrosophic domain condition. Thus, the result of the BCPNHWA aggregation remains a valid BCPNN.

3.4.2 Proposition

The BCPNHWA operator maintains the inherent bipolar structure of the input BCPNNs by preserving their positive and negative components throughout the aggregation process.

Proof. The positive components corresponding to truth, indeterminacy and falsity are aggregated through Hamacher t-norm and t-conorm operations, which preserve their non-negative

characteristics. Consequently,

$$T^{+}_{\Re(\wp_b)}, iT^{+}_{\Im(\wp_b)}, I^{+}_{\Re(\wp_b))}, iI^{+}_{\Im(\wp_b)}, F^{+}_{\Re(\wp_b)}, iF^{+}_{\Im(\wp_b)} \in [0,1]$$

For the negative components, the Hamacher aggregation procedure is initially applied to their absolute values, after which the negative sign is retained in the aggregated result. Consequently,

$$T^{-}_{\Re(\wp_b)}, iT^{-}_{\Im(\wp_b)}, I^{-}_{\Re(\wp_b))}, iI^{-}_{\Im(\wp_b)}, F^{-}_{\Re(\wp_b)}, iF^{-}_{\Im(\wp_b)} \in [-1,0]$$

Accordingly, the aggregation procedure maintains the original bipolar configuration by preserving the distinction between positive and negative components.

3.4.3 Proposition

The BCPNHWA operator ensures that the resulting real and imaginary components remain within their respective prescribed domains.

Proof. Each real and imaginary element in Equation (2) is calculated using input elements lying within the unit interval, using the Hamacher t-norm, t-conorm, weighted power and multiplication operations. These operations retain the unit interval if $s > 0$. Thus, the positive real and imaginary parts of the aggregated BCPNN are within $[0,1]$. Negative real and imaginary elements are obtained by aggregating the absolute value of these elements using the same aggregation process and adding a negative sign to them. This implies that they lie in $[-1,0]$. Thus, the complex number form of the BCPNN is maintained through aggregation.

3.4.4 Proposition

The BCPNHOWA and BCPNHHA operators are likewise well-defined aggregation mechanisms within the BCPNS framework.

Proof. The distinction between BCPNHOWA and BCPNHWA lies primarily in the ordering of the input BCPNNs and the corresponding ordered weight vector. Reordering the input elements does not alter their admissible domains, while the associated ordered weights satisfy the normalisation condition. Consequently, the closure property established for BCPNHWA extends to BCPNHOWA, ensuring that the aggregated output remains within the prescribed BCPNS domain. In contrast, BCPNHHA incorporates linked weights and a balancing coefficient to integrate the aggregation process. Under the definition of hybrid aggregation, these linked weights retain their normalised structure.

Therefore, the domain-preserving characteristics of the underlying Hamacher operations remain applicable to BCPNHHA. Both BCPNHOWA and BCPNHHA consequently maintain the positive and negative components, preserve the admissible ranges of the real and imaginary components, and retain the Pythagorean neutrosophic characteristics of the original BCPNNs. It follows that BCPNHWA, BCPNHOWA and BCPNHHA constitute well-defined aggregation operators over the BCPNS domain. Their aggregated outputs preserve the essential bipolar, complex-valued, Pythagorean and neutrosophic characteristics of the input information. These closure and domain-preservation results provide the mathematical basis for examining the subsequent properties of idempotency, monotonicity and boundedness.

3.5 Definition

Although BCPNHOWA follows the aggregation principles underlying BCPNHWA, its distinguishing feature is the requirement to arrange the input information in descending order before performing aggregation. Consider a collection of BCPNHNs.

$$\wp_{\flat}=\begin{pmatrix}\left\{T^{+}_{\Re(\wp_{\flat})}+iT^{+}_{\Im(\wp_{\flat})},T^{-}_{\Re(\wp_{\flat})}+iT^{-}_{\Im(\wp_{\flat})}\right\},\\ \left\{I^{+}_{\Re(\wp_{\flat})}+iI^{+}_{\Im(\wp_{\flat})},I^{-}_{\Re(\wp_{\flat})}+iI^{-}_{\Im(\wp_{\flat})}\right\},\\ \left\{F^{+}_{\Re(\wp_{\flat})}+iF^{+}_{\Im(\wp_{\flat})},F^{-}_{\Re(\wp_{\flat})}+iF^{-}_{\Im(\wp_{\flat})}\right\}\end{pmatrix},\flat=1,2,\cdots,\hat{\flat}$$

then

$$\mathrm{BCPNHOWA}(\wp_1,\wp_2,\cdots,\wp_{\hat{\flat}})=\bigoplus_{\flat=1}^{\hat{\flat}}\xi_{\flat}\mathfrak{B}_{o(\flat)} \tag{3}$$

The BCPNHOWA operator uses a weight vector, where $\xi=(\xi_1,\xi_2,\ldots,\xi_{\hat{\flat}})$ is a weight vector, $\xi_{\hat{\flat}}\in[0,1]$ and $\sum_{\flat=1}^{\hat{\flat}}\xi_{\flat}=1$ Furthermore, $(o(1),o(2),\ldots,o(\hat{\flat}))$ is a permutation such that $o(\flat-1)\geq o(\flat)\ \forall\ \flat=2,3,\ldots,\hat{\flat}$.

3.6 Theorem

Consider a gathering of BCPNHNs

$$\wp_{\flat}=\begin{pmatrix}\left\{T^{+}_{\Re(\wp_{\flat})}+iT^{+}_{\Im(\wp_{\flat})},T^{-}_{\Re(\wp_{\flat})}+iT^{-}_{\Im(\wp_{\flat})}\right\},\\ \left\{I^{+}_{\Re(\wp_{\flat})}+iI^{+}_{\Im(\wp_{\flat})},I^{-}_{\Re(\wp_{\flat})}+iI^{-}_{\Im(\wp_{\flat})}\right\},\\ \left\{F^{+}_{\Re(\wp_{\flat})}+iF^{+}_{\Im(\wp_{\flat})},F^{-}_{\Re(\wp_{\flat})}+iF^{-}_{\Im(\wp_{\flat})}\right\}\end{pmatrix},\flat=1,2,\cdots,\hat{\flat}$$

then aggregating this with then help of equation (3), we get

$$\mathrm{BCPNHOWA}(\wp_1,\wp_2,\cdots,\wp_{\hat{\mathfrak{b}}}) =$$

$$\left(\begin{array}{l}
\left(\dfrac{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(\mathrm{T}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)-\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{T}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(\mathrm{T}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{T}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}+\mathrm{i}\left(\dfrac{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(\mathrm{T}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)-\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{T}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(\mathrm{T}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{T}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}},\\
\left(-\dfrac{(\eth_s)^{\frac{1}{2}}\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|\mathrm{T}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})}|^{\xi_{\mathfrak{b}}}}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(1+(\mathrm{T}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2))^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|(\mathrm{T}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2|^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}+\mathrm{i}\left(-\dfrac{(\eth_s)^{\frac{1}{2}}\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|\mathrm{T}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})}|^{\xi_{\mathfrak{b}}}}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(1+(\mathrm{T}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2))^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|(\mathrm{T}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2|^{\xi_{\mathfrak{b}}}\right)}\right);\\
\left(\dfrac{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(\mathrm{I}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)-\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{I}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(\mathrm{I}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{I}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}+\mathrm{i}\left(\dfrac{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(\mathrm{I}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)-\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{I}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(\mathrm{I}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{I}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}},\\
\left(-\dfrac{(\eth_s)^{\frac{1}{2}}\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|\mathrm{I}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})}|^{\xi_{\mathfrak{b}}}}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(1+(\mathrm{I}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2))^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|(\mathrm{I}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2|^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}+\mathrm{i}\left(-\dfrac{(\eth_s)^{\frac{1}{2}}\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|\mathrm{I}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})}|^{\xi_{\mathfrak{b}}}}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(1+(\mathrm{I}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2))^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}|(\mathrm{I}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2|^{\xi_{\mathfrak{b}}}\right)}\right);\\
\left(\dfrac{(\eth_s)^{\frac{1}{2}}\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(\mathrm{F}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^{\xi_{\mathfrak{b}}}}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(1+\mathrm{F}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})}))^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(\mathrm{F}^{+}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}+\mathrm{i}\left(\dfrac{(\eth_s)^{\frac{1}{2}}\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(\mathrm{F}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^{\xi_{\mathfrak{b}}}}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)(1+\mathrm{F}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})}))^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(\mathrm{F}^{+}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^{\xi_{\mathfrak{b}}}\right)}\right),\\
\left(-\dfrac{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)|(\mathrm{F}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2|)^{\xi_{\mathfrak{b}}}\right)-\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{F}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)|(\mathrm{F}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2|)^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{F}^{-}_{\Re(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}+\mathrm{i}\left(-\dfrac{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)|(\mathrm{F}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2|)^{\xi_{\mathfrak{b}}}\right)-\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{F}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}{\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1+(\eth_s-1)|(\mathrm{F}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2|)^{\xi_{\mathfrak{b}}}\right)+(\eth_s-1)\left(\prod\limits_{\mathfrak{b}=1}^{\hat{\mathfrak{b}}}(1-(\mathrm{F}^{-}_{\Im(\wp_{\mathrm{o}(\mathfrak{b})})})^2)^{\xi_{\mathfrak{b}}}\right)}\right)^{\frac{1}{2}}
\end{array}\right)$$

The BCPNHOWA operator satisfies the fundamental mathematical properties of idempotency, boundedness and monotonicity.

3.7 Definition

Consider a gathering of BCPNHNs

$$\wp_{\flat} = \begin{pmatrix} \left\{ T^{+}_{\Re(\wp_{\flat})} + iT^{+}_{\Im(\wp_{\flat})}, T^{-}_{\Re(\wp_{\flat})} + iT^{-}_{\Im(\wp_{\flat})} \right\}, \\ \left\{ I^{+}_{\Re(\wp_{\flat})} + iI^{+}_{\Im(\wp_{\flat})}, I^{-}_{\Re(\wp_{\flat})} + iI^{-}_{\Im(\wp_{\flat})} \right\}, \\ \left\{ F^{+}_{\Re(\wp_{\flat})} + iF^{+}_{\Im(\wp_{\flat})}, F^{-}_{\Re(\wp_{\flat})} + iF^{-}_{\Im(\wp_{\flat})} \right\} \end{pmatrix}, \flat = 1,2,\cdots,\hat{\flat}$$

then

$$\mathrm{BCPNHHA}(\wp_1, \wp_2, \cdots, \wp_{\hat{\flat}}) = \bigoplus_{\flat=1}^{\hat{\flat}} \theta_{\flat} \mathfrak{B}^{*}_{\flat}$$

The BCPNHHA operator uses a linked weight vector, where $\theta = (\theta_1, \theta_2, \ldots, \theta_{\hat{\flat}})$ is a linked weight vector, $\theta_{\hat{\flat}} \in [0,1]$ and $\sum_{\flat=1}^{\hat{\flat}} \theta_{\flat} = 1$, $\mathfrak{B}^{*}_{\flat}(\mathfrak{B}^{*}_{\flat} = (\hat{\flat}\xi_{\hat{\flat}}\wp_{\flat})$, $\hat{\flat}$ is the balancing coefficient.

3.8 Theorem

Consider a gathering of BCPNHNs

$$\wp_{\flat} = \begin{pmatrix} \left\{ T^{+}_{\Re(\wp_{\flat})} + iT^{+}_{\Im(\wp_{\flat})}, T^{-}_{\Re(\wp_{\flat})} + iT^{-}_{\Im(\wp_{\flat})} \right\}, \\ \left\{ I^{+}_{\Re(\wp_{\flat})} + iI^{+}_{\Im(\wp_{\flat})}, I^{-}_{\Re(\wp_{\flat})} + iI^{-}_{\Im(\wp_{\flat})} \right\}, \\ \left\{ F^{+}_{\Re(\wp_{\flat})} + iF^{+}_{\Im(\wp_{\flat})}, F^{-}_{\Re(\wp_{\flat})} + iF^{-}_{\Im(\wp_{\flat})} \right\} \end{pmatrix}, \flat = 1,2,\cdots,\hat{\flat}$$

then aggregating this with then help of equation (5), we get

$$\mathrm{BCPNHOWA}(\wp_1, \wp_2, \cdots, \wp_{\hat{\flat}}) =$$

$$\left(\begin{array}{l}
\left(\dfrac{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(T^{+}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right) - \left(\prod_{\flat=1}^{\hat{\flat}} (1-(T^{+}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(T^{+}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} (1-(T^{+}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}} + \mathrm{i} \left(\dfrac{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(T^{+}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right) - \left(\prod_{\flat=1}^{\hat{\flat}} (1-(T^{+}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(T^{+}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} (1-(T^{+}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}}, \\
\left(- \dfrac{(\delta_s)^{\frac{1}{2}} \prod_{\flat=1}^{\hat{\flat}} |T^{-}_{\Re(\wp_{O(\flat)})}|^{\xi_{\flat}}}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(1+(T^{-}_{\Re(\wp_{O(\flat)})})^2))^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} |(T^{-}_{\Re(\wp_{O(\flat)})})^2|^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}} + \mathrm{i} \left(- \dfrac{(\delta_s)^{\frac{1}{2}} \prod_{\flat=1}^{\hat{\flat}} |T^{-}_{\Im(\wp_{O(\flat)})}|^{\xi_{\flat}}}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(1+(T^{-}_{\Im(\wp_{O(\flat)})})^2))^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} |(T^{-}_{\Im(\wp_{O(\flat)})})^2|^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}}; \\
\left(\dfrac{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(I^{+}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right) - \left(\prod_{\flat=1}^{\hat{\flat}} (1-(I^{+}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(I^{+}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} (1-(I^{+}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}} + \mathrm{i} \left(\dfrac{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(I^{+}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right) - \left(\prod_{\flat=1}^{\hat{\flat}} (1-(I^{+}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(I^{+}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} (1-(I^{+}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}}, \\
\left(- \dfrac{(\delta_s)^{\frac{1}{2}} \prod_{\flat=1}^{\hat{\flat}} |I^{-}_{\Re(\wp_{O(\flat)})}|^{\xi_{\flat}}}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(1+(I^{-}_{\Re(\wp_{O(\flat)})})^2))^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} |(I^{-}_{\Re(\wp_{O(\flat)})})^2|^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}} + \mathrm{i} \left(- \dfrac{(\delta_s)^{\frac{1}{2}} \prod_{\flat=1}^{\hat{\flat}} |I^{-}_{\Im(\wp_{O(\flat)})}|^{\xi_{\flat}}}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(1+(I^{-}_{\Im(\wp_{O(\flat)})})^2))^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} |(I^{-}_{\Im(\wp_{O(\flat)})})^2|^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}}; \\
\left(\dfrac{(\delta_s)^{\frac{1}{2}} \prod_{\flat=1}^{\hat{\flat}} (F^{+}_{\Re(\wp_{O(\flat)})})^{\xi_{\flat}}}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(1+F^{+}_{\Re(\wp_{O(\flat)})}))^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} (F^{+}_{\Re(\wp_{O(\flat)})})^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}} + \mathrm{i} \left(\dfrac{(\delta_s)^{\frac{1}{2}} \prod_{\flat=1}^{\hat{\flat}} (F^{+}_{\Im(\wp_{O(\flat)})})^{\xi_{\flat}}}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)(1+F^{+}_{\Im(\wp_{O(\flat)})}))^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} (F^{+}_{\Im(\wp_{O(\flat)})})^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}}, \\
\left(- \dfrac{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)|(F^{-}_{\Re(\wp_{O(\flat)})})^2|)^{\xi_{\flat}} \right) - \left(\prod_{\flat=1}^{\hat{\flat}} (1-(F^{-}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)|(F^{-}_{\Re(\wp_{O(\flat)})})^2|)^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} (1-(F^{-}_{\Re(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}} + \mathrm{i} \left(- \dfrac{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)|(F^{-}_{\Im(\wp_{O(\flat)})})^2|)^{\xi_{\flat}} \right) - \left(\prod_{\flat=1}^{\hat{\flat}} (1-(F^{-}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)}{\left(\prod_{\flat=1}^{\hat{\flat}} (1+(\delta_s-1)|(F^{-}_{\Im(\wp_{O(\flat)})})^2|)^{\xi_{\flat}} \right) + (\delta_s-1) \left(\prod_{\flat=1}^{\hat{\flat}} (1-(F^{-}_{\Im(\wp_{O(\flat)})})^2)^{\xi_{\flat}} \right)} \right)^{\frac{1}{2}}
\end{array} \right)$$

The BCPNHOWA operator satisfies the fundamental mathematical properties of idempotency, boundedness and monotonicity.

The BCPNHHA operator satisfies the essential mathematical properties of idempotency, boundedness and monotonicity.

4. APPLICATION

The proposed MCDM framework is applied to an educational technology selection problem to demonstrate the operational applicability of the developed HAOs. Rather than constituting an empirical investigation based on an educational dataset, the numerical example serves as an illustrative decision-making scenario through which the proposed methodology can be systematically examined. Three educational technology alternatives are evaluated against three criteria to demonstrate the complete implementation procedure within the BCPNS framework.

Educational technology selection in practical settings frequently involves ambiguous judgements, incomplete information, hesitation and conflicting evaluations, including simultaneous favourable and unfavourable assessments of the available alternatives. To accommodate these complexities, the proposed framework integrates bipolar representations, complex-valued information, Pythagorean restrictions and independent neutrosophic truth, indeterminacy and falsity degrees within a unified assessment structure. The developed HAOs subsequently synthesise criterion-specific evaluations to generate an overall assessment for each candidate technology. Through this illustrative application, the framework demonstrates its capacity to support structured comparisons and establish preference rankings under uncertain decision-making conditions. The computational procedure underlying the proposed MCDM methodology is detailed in the following algorithm, while its corresponding graphical representation is provided in Fig. 1.

4.1 Algorithm

Step 1: Identify and establish the set of candidate alternatives and the corresponding evaluation criteria relevant to the educational technology selection problem.

Step 2: Obtain the assessments of each alternative with respect to the specified criteria and represent the resulting evaluation information using BCPNNs.

Step 3: Organise the collected assessments into a BCPNS decision matrix, ensuring that the evaluation information for each alternative is appropriately associated with its corresponding criteria.

Step 4: Determine suitable weights for the evaluation criteria and employ the proposed BCPNS-HAO to synthesise the criterion-specific assessments associated with each alternative.

Step 5: Calculate the score of each alternative by applying the appropriate BCPNN score function to its aggregated evaluation. This transformation converts complex neutrosophic information into a comparable real-valued measure while jointly accounting for truth, indeterminacy and falsity components and their associated positive and negative assessments.

Step 6: Establish the preference ordering of the alternatives by arranging their calculated scores in descending order.

Step 7: Identify the alternative attaining the highest score as the most preferable educational technology. Subsequently, interpret the resulting ranking to determine the relative suitability of the evaluated technologies for implementation within the educational setting.

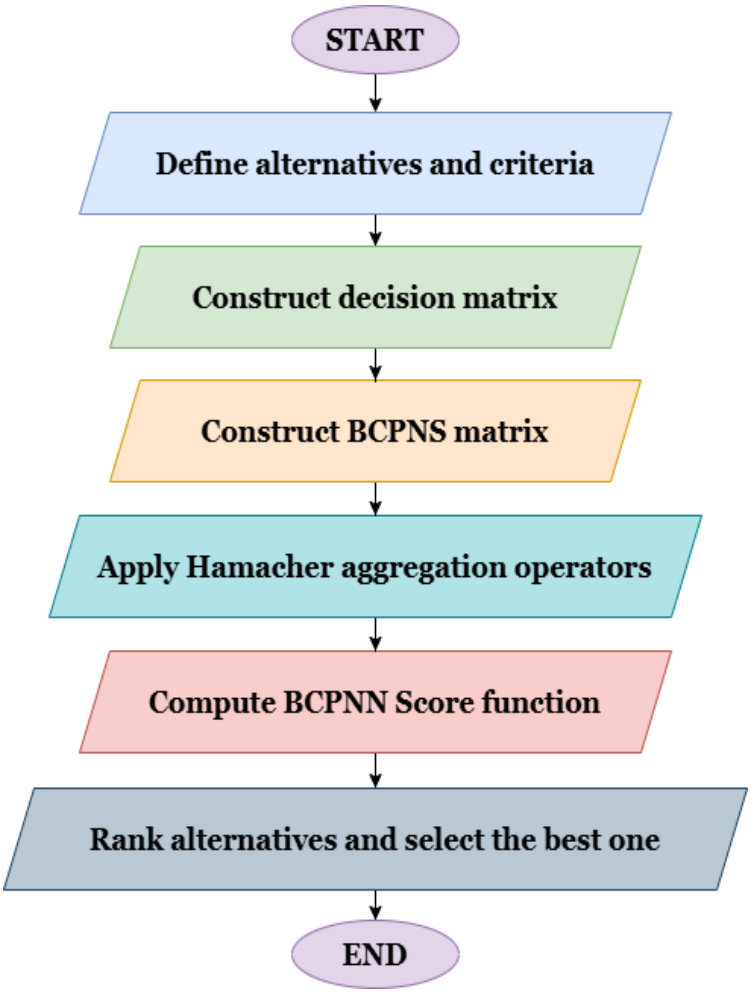


Fig. 1: Graphical Algorithm

4.2 Problem Description

The increasing adoption of advanced digital technologies in school education has introduced new possibilities for enhancing instructional practices and student learning outcomes. Nevertheless, identifying the most appropriate technological solution remains a challenging decision-making task, as available alternatives differ in their educational effectiveness, capacity to promote student engagement, affordability and ease of use. These variations necessitate a structured MCDM framework capable of systematically assessing competing technologies against multiple evaluation criteria and establishing their relative preference rankings.

In this study, three educational technology alternatives are considered, namely AI-based personalized learning ($\mathfrak{Y}_1$), smart classroom system ($\mathfrak{Y}_2$), and AI-based adaptive assessment ($\mathfrak{Y}_3$). These alternatives are evaluated with respect to three important criteria: learning effectiveness (C_1), student engagement (C_2), and implementation cost (C_3). Among these criteria, C_1 and C_2 are considered benefit criteria, whereas C_3 is considered a cost criterion. Each alternative is assessed by the decision-maker against the three specified evaluation criteria. Given that educational technology selection may be characterised by incomplete knowledge, hesitant judgements, contradictory evidence and simultaneous favourable and unfavourable evaluations, the corresponding membership information is expressed through BCPNNs, as presented in Table 2. The proposed HAOs are subsequently applied within the BCPNS framework to synthesise the criterion-specific assessments and derive a comprehensive evaluation of each alternative.

Table 2
Illustrative Conversion of Expert Judgments into BCPNS Components

Expert Judgment	Practical Interpretation	Possible BCPNS Representation
High positive evidence	Strong evidence that the technology improves the criterion	High positive membership
Moderate positive evidence	Some evidence of improvement, but not consistently strong	Moderate positive membership
High negative evidence	Strong evidence that the technology performs poorly	High magnitude of negative membership
High confidence in evidence	Evidence is reliable and consistent	High truth degree
Conflicting or insufficient evidence	Evidence is incomplete, inconsistent, or uncertain	High indeterminacy degree
Evidence contradicting the assessment	Available evidence does not support the positive judgment	High falsity degree
Stable performance across contexts	Assessment remains similar across learner groups or periods	Low phase variation
Context-dependent performance	Assessment varies with learner group, period, or implementation condition	Different phase/context value

Consider an MCDM problem for selecting the most suitable educational technology for a school. The set of alternatives and criteria can be represented as $A = \{\mathfrak{Ə}_1, \mathfrak{Ə}_2, \mathfrak{Ə}_3\}$ and $C = \{\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3\}$ where $\mathfrak{Ə}_1$ = AI-based Personalized Learning; $\mathfrak{Ə}_2$ = Smart Classroom System; $\mathfrak{Ə}_3$ = AI-based Adaptive Assessment and C_1 = Learning Effectiveness; C_2 = Student Engagement; C_3 = Implementation Cost. The corresponding criterion-weight vector is assumed to be

$$\omega = (\omega_1, \omega_2, \omega_3) = (.400, .333, .267)$$

where

$$\sum_{j=1}^{3} \omega_j = 1.$$

The positional weight vector $\theta = (\theta_1, \ldots, \theta_{\hat{\mathfrak{b}}})$ of the BCPNHHA operator is determined using the normal-distribution-based method. Let the weighted arguments be arranged in descending order of their score values. The mean and standard deviation of the position indices are

$$\mu = (\hat{\mathfrak{b}} + 1)/2, \sigma = \sqrt{\left(\frac{1}{\hat{\mathfrak{b}}}\right) \Sigma_{\{j=1\}}^{\{\hat{\mathfrak{b}}\}} (j - \mu)^2},$$

and the weights are generated as

$$\theta_j = exp\left(-\frac{(j - \mu)^2}{(2\sigma^2)}\right) \Sigma_{\{k=1\}}^{\{\hat{\mathfrak{b}}\}} exp\left(-\frac{(k - \mu)^2}{(2\sigma^2)}\right), j = 1, 2, \ldots, \hat{\mathfrak{b}}.$$

This gives $\theta_j \in [0,1]$ and $\Sigma\theta_j = 1$. The scheme assigns the largest weight to the middle-ranked value and smaller weights to the extreme ones, which reduces the influence of unduly high or low assessments. For $\hat{\mathfrak{b}} = 3$, we obtain $\mu = 2, \sigma = \sqrt{(2/3)}$, and $\theta = (\theta_1, \theta_2, \theta_3) = (.2429, .5142, .2429)$.

Table 3 presents the BCPNS decision matrix, which systematically organises the evaluations of the three alternatives against the three specified criteria.

Table 3
Decision Matrix

Alternatives	C_1	C_2	C_3
$\mathfrak{H}_1$	T(.4 + i.1 − .5 − i.7) I(.2 + i.7 − .1 − i.3) F(.5 + i.3 − .2 − i.5)	T(.4 + i.3 − .2 − i.1) I(.5 + i.4 − .3 − i.2) F(.6 + i.8 − .1 − i.2)	T(.3 + i.5 − .7 − i.9) I(.9 + i.8 − .5 − i.4) F(.6 + i.7 − .1 − i.5)
$\mathfrak{H}_2$	T(.8 + i.7 − .1 − i.2) I(.7 + i.3 − .5 − i.4) F(.1 + i.2 − .3 − i.4)	T(.5 + i.7 − .8 − i.1) I(.8 + i.7 − .4 − i.5) F(.9 + i.2 − .3 − i.1)	T(.2 + i.5 − .7 − i.4) I(.2 + i.3 − .5 − i.9) F(.3 + i.2 − .5 − i.1)
$\mathfrak{H}_3$	T(.9 + i.8 − .2 − i.3) I(.8 + i.7 − .3 − i.4) F(.7 + i.6 − .5 − i.4)	T(.7 + i.6 − .5 − i.4) I(.6 + i.5 − .4 − i.3) F(.5 + i.4 − .3 − i.2)	T(.7 + i.9 − .5 − i.3) I(.8 + i.6 − .4 − i.2) F(.7 + i.5 − .2 − i.1)

The aggregated decision matrix is obtained by applying the three proposed operators, BCPNHWA, BCPNHOWA and BCPNHHA, as defined in Equations (2), (4) and (6), respectively, to synthesise the criterion-specific evaluations of each alternative presented in the decision matrix in Table 3.

Table 4
Aggregated Matrix for Alternatives using BCPNHWA, BCPNHOWA and BCPNHHA Operators

Alternatives	BCPNHWA	BCPNHOWA	BCPNHHA
$\mathfrak{H}_1$	T(.376 + i.325 − .403 − i.391) I(.625 + i.671 − .221 − i.283) F(.557 + i.521 − .148 − i.430)	T(.377 + i.326 − .403 − i.392) I(.653 + i.672 − .222 − i.283) F(.558 + i.521 − .149 − i.431)	T(.383 + i.319, −.333 − i.271), I(.619 + i.620, −.241 − i.258), F(.569 + i.586, −.137 − i.385)
$\mathfrak{H}_2$	T(.634 + i.659 − .336 − i.191) I(.680 + i.499 − .464 − i.534) F(.278 + i.200 − .368 − i.270)	T(.627 + i.648 − .333 − i.209) I(.653 + i.469 − .471 − i.556) F(.259 + i.200 − .383 − i.270)	T(.565 + i.616, −.418 − i.245), I(.608 + i.453, −.492 − i.620), F(.303 + i.218, −.394 − i.236)
$\mathfrak{H}_3$	T(.810 + i.795 − .346 − i.330) I(.750 + i.619 − .356 − i.302) F(.625 + i.499 − .381 − i.287)	T(.780 + i.796 − .391 − i.337) I(.739 + i.598 − .370 − i.283) F(.612 + i.480 − .347 − i.253)	T(.777 + i.815, −.397 − i.343), I(.750 + i.602, −.386 − i.295), F(.658 + i.519, −.341 − i.251)

The score values of the aggregated alternatives are subsequently calculated to establish their relative preference rankings.

Table 5
Overall Score Function for Ranking

Alternatives	BCPNHWA	BCPNHOWA	BCPNHHA
$\mathfrak{H}_1$	.3972	.3972	.3923
$\mathfrak{H}_2$	.4539	.4552	.4486
$\mathfrak{H}_3$	.4569	.4664	.4585

The numerical illustration demonstrates the applicability of the proposed BCPNS-based Hamacher aggregation framework to educational technology selection under uncertain decision-making conditions. To obtain comprehensive assessments of the three candidate alternatives, BCPNHWA, BCPNHOWA and BCPNHHA are applied to synthesise their evaluations across the three specified criteria. The resulting aggregated BCPNS information is reported in Table 4, providing the basis for subsequent comparative assessment.

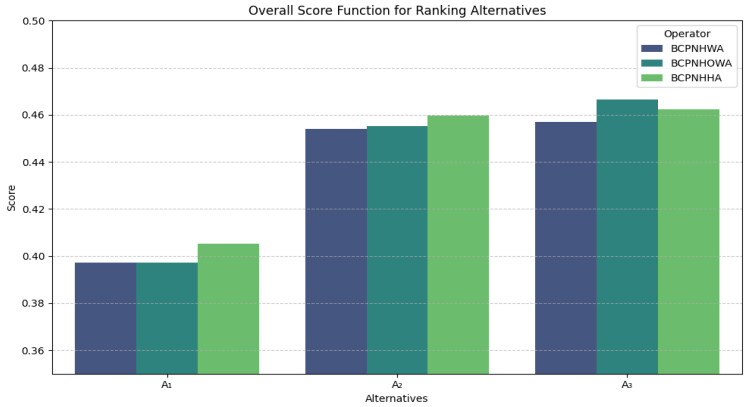


Fig. 2: Ranking of Overall Score Values

These aggregated evaluations are then converted into comparable numerical scores through the proposed score function, with the corresponding results presented in Table 5. Under the BCPNHWA operator, the scores obtained for $\mathfrak{H}_1, \mathfrak{H}_2$ and $\mathfrak{H}_3$are .3972, .4539, and .4569, respectively. Under the BCPNHOWA operator, the corresponding scores are .3972, .4552, and .4664, while under the BCPNHHA operator they are .3923, .4486, and .4585. Although the numerical values differ slightly from one operator to another, all three operators produce the same ranking of the alternatives, $\mathfrak{H}_3 > \mathfrak{H}_2 > \mathfrak{H}_1$ and the visual representation of this ranking is presented in Fig. 2.

Therefore, $\mathfrak{H}_3$, namely the AI-based adaptive assessment, is identified as the most suitable educational technology among the considered alternatives, followed by the smart classroom system ($\mathfrak{H}_2$), while the AI-based personalized learning ($\mathfrak{H}_1$) is ranked last. The identical preference ordering obtained through BCPNHWA, BCPNHOWA and BCPNHHA indicates that the resulting ranking remains consistent across the three aggregation mechanisms. This agreement suggests that, within the numerical example considered, the selection outcome is not sensitive to the choice of aggregation operator. Furthermore, the findings illustrate the capacity of the proposed operators to synthesise multidimensional bipolar complex neutrosophic assessments and generate a systematic preference ranking of educational technology alternatives.

Nevertheless, the numerical application is intended exclusively as an illustrative demonstration rather than an empirical investigation of educational technology performance. Its primary purpose is to clarify the computational implementation and practical applicability of the proposed BCPNS-based HAOs. The selection of three alternatives and three evaluation criteria provide a manageable decision-making scenario through which the aggregation and ranking procedures can be examined transparently. The results generated by BCPNHWA, BCPNHOWA and BCPNHHA are compared to assess the consistency of their respective preference orderings, while the sensitivity analysis presented in Section 4.4 investigates how variations in criterion weights influence the resulting rankings. Together, these assessments provide additional insight into the consistency of the proposed methodology within the specified numerical setting. However, the illustrative results do not establish the empirical superiority of the proposed framework over existing MCDM methodologies. Further investigations should therefore incorporate larger empirical datasets, assessments from multiple experts, and more extensive sets of evaluation criteria and candidate alternatives.

4.3 Comparative Analysis

A systematic comparison with established aggregation methodologies is undertaken to examine the distinctive mathematical and representational characteristics of the proposed BCPNS-based

Hamacher aggregation framework. The comparative assessment considers the informational structures supported by existing approaches, with particular emphasis on their ability to accommodate bipolar evaluations, complex-valued membership information, Pythagorean restrictions and independently represented neutrosophic truth, indeterminacy and falsity components. The underlying aggregation mechanisms are also examined to identify differences in their mathematical formulations and decision-making capabilities. The selected methodologies encompass bipolar fuzzy Dombi aggregation operators, generalised neutrosophic HAOs, bipolar fuzzy HAOs, complex fuzzy aggregation operators, generalised aggregation mechanisms for complex single-valued neutrosophic hesitant fuzzy information, bipolar complex fuzzy HAOs, fractional fuzzy HAOs and hesitant bipolar complex fuzzy Hamacher power aggregation operators.

Bipolar fuzzy Dombi aggregation mechanisms have been developed to facilitate decision-making through weighted, ordered weighted, hybrid and geometric aggregation procedures [14]. These operators provide flexible mathematical tools for integrating favourable and unfavourable assessments within bipolar fuzzy environments. Nevertheless, their representational structure does not simultaneously incorporate complex-valued membership information, Pythagorean constraints and independent neutrosophic truth, indeterminacy and falsity components. The proposed BCPNS framework extends this informational scope by integrating these characteristics within a unified mathematical representation. Generalised neutrosophic number HAOs have also been formulated for group decision-making applications, providing mechanisms for synthesising assessments involving truth, indeterminacy and falsity information [18]. Although this approach offers considerable flexibility in processing neutrosophic uncertainty, its mathematical structure does not jointly accommodate bipolar evaluations, complex-valued membership degrees and Pythagorean restrictions. In comparison, the proposed BCPNS-based HAOs incorporate these additional informational dimensions while retaining the independent representation of neutrosophic components.

Another relevant methodological development concerns bipolar fuzzy HAOs designed for MADM problems [26]. These operators employ Hamacher aggregation mechanisms to process positive and negative evaluations, thereby accommodating the bipolar nature of uncertain decision information. However, their underlying formulation does not simultaneously represent complex-valued membership information, Pythagorean conditions and independently specified neutrosophic truth, indeterminacy and falsity assessments. The proposed framework addresses this representational limitation through the integrated mathematical structure of BCPNSs. Complex fuzzy aggregation has been investigated through the development of arithmetic aggregation operators and the examination of mathematical relationships between complex fuzzy representations and Pythagorean fuzzy numbers [6]. These developments facilitate the manipulation of complex-valued membership information and establish relevant mathematical properties of the corresponding aggregation mechanisms. Nevertheless, the framework does not simultaneously incorporate bipolar assessments and independent neutrosophic truth, indeterminacy and falsity components. The BCPNS representation adopted in the present investigation accommodates these characteristics within a common information structure.

Further developments have introduced generalised aggregation operators for complex single-valued neutrosophic hesitant fuzzy numbers, enabling a more detailed representation of uncertainty and hesitation in decision-making environments [12]. Although these mechanisms provide enhanced flexibility in processing complex neutrosophic information, they do not explicitly integrate bipolar and Pythagorean characteristics within the same mathematical formulation. Consequently, their informational scope differs from that of the proposed BCPNS framework, which simultaneously

incorporates these additional dimensions. Among the existing methodologies, bipolar complex fuzzy HAOs exhibit particularly close conceptual and mathematical similarities to the proposed approach [19]. Both frameworks integrate bipolar assessments, complex-valued membership information and Hamacher aggregation procedures. However, the existing bipolar complex fuzzy formulation does not explicitly incorporate Pythagorean membership restrictions alongside independently represented neutrosophic truth, indeterminacy and falsity components. The proposed BCPNS-based HAOs extend this structure by introducing these additional informational characteristics into the aggregation process.

Fractional fuzzy HAOs have also been developed and applied to decision-making problems through extended TOPSIS, TODIM and GRA methodologies [1]. These operators provide alternative mechanisms for aggregating uncertain information within fractional fuzzy environments. Nevertheless, their mathematical representation does not simultaneously accommodate bipolar evaluations, complex-valued membership degrees, Pythagorean restrictions and independent neutrosophic components. The proposed BCPNS framework differs by integrating these characteristics into a unified aggregation structure. Hesitant bipolar complex fuzzy Hamacher power aggregation operators have additionally been introduced for AI-based energy management applications [11]. This approach combines bipolar complex fuzzy representations with hesitant information and Hamacher power aggregation mechanisms, providing a flexible means of processing uncertain evaluations. Despite these capabilities, the corresponding mathematical framework does not explicitly incorporate Pythagorean restrictions together with independently represented neutrosophic truth, indeterminacy and falsity components. These additional informational dimensions distinguish the proposed BCPNS-based aggregation methodology from the existing hesitant bipolar complex fuzzy approach.

Table 6 presents a consolidated comparison of the proposed framework with the selected aggregation methodologies, highlighting differences in their underlying information structures and representational capabilities. Bipolar fuzzy Hamacher aggregation approaches primarily accommodate favourable and unfavourable assessments, whereas complex fuzzy aggregation methods represent uncertainty through complex-valued membership functions. Neutrosophic Hamacher aggregation frameworks independently incorporate truth, indeterminacy and falsity information, while bipolar complex fuzzy HAOs combine bipolar evaluations with complex-valued membership representations. Although these methodologies provide distinct mechanisms for modelling and aggregating uncertain information, none of the compared frameworks simultaneously incorporates the complete combination of bipolarity, complex-valued membership information, Pythagorean restrictions and independent neutrosophic components considered in the present investigation.

In contrast, the proposed BCPNS-based Hamacher framework consolidates these informational characteristics within a single mathematical decision-making environment. Its integrated structure accommodates positive and negative judgements, complex-valued membership information, Pythagorean constraints and independently specified truth, indeterminacy and falsity assessments, thereby enabling a more comprehensive representation of uncertainty when these characteristics coexist. The corresponding HAOs provide a systematic mechanism for aggregating such multidimensional information, particularly in educational technology evaluation problems involving ambiguous, incomplete or conflicting assessments. Importantly, the methodological contribution extends beyond the number of informational characteristics identified in Table 6. Its principal novelty resides in the specific mathematical formulation of the BCPNS information space and the development of compatible Hamacher aggregation mechanisms capable of processing information

within that structure.

Table 6
Comparison of the Proposed BCPNS Hamacher Aggregation Framework with Representative Existing Models

Model	Ref.	Bipolar	Complex	Pythagorean	Neutrosophic T/I/F	Hamacher
Bipolar Fuzzy Dombi Aggregation	[14]	✓	×	×	×	×
Generalized Neutrosophic Hamacher Aggregation	[18]	×	×	×	✓	✓
Bipolar Fuzzy Hamacher Aggregation	[26]	✓	×	×	×	✓
Complex Fuzzy Aggregation	[6]	×	✓	×	×	×
Complex Single-Valued Neutrosophic Hesitant Fuzzy Aggregation	[12]	×	✓	×	✓	✓
Bipolar Complex Fuzzy Hamacher Aggregation	[19]	✓	✓	×	×	✓
Fractional Fuzzy Hamacher Aggregation	[1]	×	×	×	×	✓
Hesitant Bipolar Complex Fuzzy Hamacher Power Aggregation	[11]	✓	✓	×	×	✓

In contrast, the proposed BCPNS-based Hamacher framework unifies these distinct informational characteristics within a single decision-making structure. By simultaneously accommodating favourable and unfavourable judgements, complex-valued membership information, Pythagorean restrictions and independently specified truth, indeterminacy and falsity components, the BCPNS representation enables a more comprehensive characterisation of uncertainty in decision-making environments where these features coexist. The corresponding HAOs provide an integrated mathematical mechanism for processing such multidimensional information, particularly when educational technology assessments involve ambiguity, inconsistency and conflicting evaluations. Importantly, the methodological novelty does not arise merely from the broader range of characteristics identified in Table 6, but from the specific mathematical construction of the BCPNS information space and the formulation of compatible Hamacher aggregation procedures within that structure.

4.4 Analysis

The initial criterion weights reflect the decision-maker's subjective preference structure rather than objectively measured characteristics of an empirical dataset. Consequently, examining the influence of variations in these weights is essential for determining the extent to which the resulting preference ordering depends on the adopted weighting configuration. The sensitivity analysis therefore serves two complementary purposes: assessing the robustness of the obtained rankings under alternative criterion-weight distributions and investigating the dependence of the decision outcome on subjective weighting assumptions. Attention is directed towards determining whether the initially preferred educational technology maintains its relative competitiveness when the importance assigned to individual evaluation criteria changes.

Given that the criterion weights $\omega = (\omega_1, \omega_2, \omega_3)$ are subjectively assigned by the decision maker, it becomes necessary to explore whether the ranking order $\mathfrak{D}_3 > \mathfrak{D}_2 > \mathfrak{D}_1$, which is achieved, is consistent regardless of changes in the values of the weights. To this end, a sensitivity analysis of the results is carried out via a Monte Carlo simulation process where the criterion weights are resampled and the entire procedure of BCPNHWA aggregation.

4.4.1 Simulation Design

A weight vector satisfying $\omega_j \geq 0$ and $\sum_{j=1}^{3} \omega_j = 1$ was randomly generated using a Dirichlet distribution,

$$\omega^{(r)} \sim Dir(\alpha_1, \alpha_2, \alpha_3), \qquad r = 1,2, \dots, N,$$

where the concentration parameters α_j control the spread of the sampled weights around a reference point. Three sampling scenarios were considered, each with $N = 20{,}000$ independent trials, giving $60{,}000$ Monte Carlo evaluations in total:

Scenario 1 (Fully Random Weights)

$\alpha = (1,1,1)$, which samples $\omega^{(r)}$ uniformly over the entire weight simplex, representing the case of complete uncertainty about the relative importance of the criteria.

Scenario 2 (Moderate Perturbation)

$\alpha = 40 \times (.400, .333, .267)$, which concentrates the sampled weights around the baseline weight vector while still allowing moderate deviation.

Scenario 3 (Mild Perturbation)

$\alpha = 150 \times (.400, .333, .267)$, which produces a tighter spread around the baseline vector, simulating small adjustments a decision-maker might realistically make.

For each sampled weight vector $\omega^{(r)}$, the BCPNHWA operator (Equation (2)) was applied to get the aggregated BCPNN of each alternative, the score function was calculated, and alternatives were ranked based on their scores in descending order. The number of times each alternative was ranked as the highest ranked alternative, along with its average score value and standard deviation of scores in all trials, was calculated. Additionally, the one-at-a-time (OAT) sweep analysis, which involved setting the value of each criterion weight from $.05$ to $.90$ with other two weights being proportionally scaled according to their base ratio, was conducted.

4.4.2 Results

Table 7 provides a consolidated statistical summary for each of the three sampling scenarios, reporting the percentage of trials in which each alternative attained the highest ranking, alongside the corresponding mean score, standard deviation and score range.

Table 7
Monte Carlo Sensitivity Analysis: Overall Percentage of Trials Ranked First ($N = 20{,}000$ per Scenario)

Scenario	Alternative	Mean Score	Std. Dev.	Score Range	% Ranked 1st
Scenario 1: Fully Random	A_1	0.3967	0.0263	[0.3432, 0.4705]	0.00%
	A_2	0.4516	0.0148	[0.4094, 0.4907]	36.13%
	A_3	0.4600	0.0169	[0.4118, 0.4889]	63.88%
Scenario 2: Moderate Perturbation	A_1	0.3980	0.0085	[0.3657, 0.4312]	0.00%
	A_2	0.4540	0.0049	[0.4350, 0.4738]	39.07%
	A_3	0.4568	0.0056	[0.4345, 0.4775]	60.94%

Scenario 3: Mild Perturbation	A_1	0.3974	0.0044	[0.3790, 0.4151]	0.00%
	A_2	0.4539	0.0026	[0.4445, 0.4652]	29.42%
	A_3	0.4569	0.0029	[0.4445, 0.4682]	70.59%

Table 8 additionally presents the distribution of complete preference orderings, indicating the frequency with which each distinct ranking sequence occurred across the simulation trials.

Table 8
Frequency of Observed Ranking Orders Across Monte Carlo Trials

Scenario	Ranking Order	Frequency (%)
Scenario 1: Fully Random	$A_3 > A_2 > A_1$	63.88%
	$A_2 > A_3 > A_1$	27.23%
	$A_2 > A_1 > A_3$	8.90%
Total		**100.00%**
Scenario 2: Moderate Perturbation	$A_3 > A_2 > A_1$	60.94%
	$A_2 > A_3 > A_1$	39.07%
Total		**100.00%**
Scenario 3: Mild Perturbation	$A_3 > A_2 > A_1$	70.59%
	$A_2 > A_3 > A_1$	29.42%
Total		**100.00%**

4.4.3 Discussion

The findings presented in Table 7 reveal that alternative $\mathfrak{D}_1$ (AI-based personalised learning) did not attain the highest ranking in any of the 20,000 Monte Carlo simulation trials conducted under the three sampling scenarios. This result indicates that $\mathfrak{D}_1$ exhibits limited competitiveness for the highest preference position across the examined variations in criterion weights. Conversely, the leading position is shared between $\mathfrak{D}_2$ (smart classroom system) and $\mathfrak{D}_3$ (AI-based adaptive assessment), with their relative preferences varying according to the sampling conditions. Under fully random sampling, $\mathfrak{D}_3$ achieves the highest ranking in 63.88% of trials, compared with 36.13% for $\mathfrak{D}_2$. A similar distribution emerges under moderate perturbation, where $\mathfrak{D}_3$ occupies the leading position in 60.94% of trials, while $\mathfrak{D}_2$ achieves this position in 39.07%. Under mild perturbation, the proportion of trials favouring $\mathfrak{D}_3$ increases to 70.59%, whereas the remaining 29.42% favour $\mathfrak{D}_2$.

The complete ranking distributions reported in Table 8 further clarify the influence of criterion-weight variations on the relative ordering of the alternatives. Within each sampling scenario, the reported ranking frequencies collectively account for 100% of the simulation trials. Scenarios 2 and 3 produce only two distinct preference sequences, namely $\mathfrak{D}_3 > \mathfrak{D}_2 > \mathfrak{D}_1$ and $\mathfrak{D}_2 > \mathfrak{D}_3 > \mathfrak{D}_1$, indicating that the principal variation concerns the interchange of the two leading alternatives. However, Scenario 1 additionally produces the ordering $\mathfrak{D}_2 > \mathfrak{D}_1 > \mathfrak{D}_3$ in 8.90% of trials, reflecting the broader distribution of criterion weights permitted under fully random sampling. Consequently, although $\mathfrak{D}_1$ never achieves the highest position, its relative placement is not entirely invariant across all scenarios, as it occasionally ranks above $\mathfrak{D}_3$ under the most extensive weight variations.

The one-at-a-time (OAT) sensitivity analysis provides further insight into the conditions governing changes in the leading alternative. Specifically, the transition between $\mathfrak{D}_2$ and $\mathfrak{D}_3$ is predominantly associated with variations in the weight assigned to C_1 (learning effectiveness). When the weight of C_1 exceeds approximately 0.50, $\mathfrak{D}_2$ becomes the highest-ranked alternative; otherwise, $\mathfrak{D}_3$ generally retains the leading position. Collectively, these findings indicate that $\mathfrak{D}_1$ consistently remains outside the highest preference position, whereas the relative superiority of $\mathfrak{D}_2$ and $\mathfrak{D}_3$ demonstrates

moderate sensitivity to the importance attributed to learning effectiveness. Accordingly, careful specification of criterion weights is essential, particularly when individual decision-makers or expert groups determine the relative significance of the evaluation criteria.

5. CONCLUSION

This investigation has developed an integrated MCDM methodology based on BCPNSs and HAOs to address decision-making environments characterised by uncertainty, incomplete information, ambiguity and conflicting assessments. The proposed framework provides a unified mathematical representation that simultaneously incorporates positive and negative evaluations, complex-valued membership information, Pythagorean restrictions and independently specified neutrosophic truth, indeterminacy and falsity components. Within this structure, three aggregation operators, namely BCPNHWA, BCPNHOWA and BCPNHHA, have been formulated to facilitate the systematic synthesis of multidimensional decision information. Their fundamental mathematical characteristics, including idempotency, monotonicity and boundedness, have also been examined to establish the theoretical properties of the proposed aggregation mechanisms. The computational applicability of the developed framework has been demonstrated through an illustrative educational technology selection problem involving three candidate alternatives and three evaluation criteria. The numerical procedure employs BCPNHWA to aggregate criterion-specific assessments, followed by the proposed score function to determine the relative preference ordering. The resulting scores are 0.3972, 0.4539 and 0.4569 for $Ɗ_1$, $Ɗ_2$ and $Ɗ_3$, respectively, producing the ranking $Ɗ_3 > Ɗ_2 > Ɗ_1$. Accordingly, $Ɗ_3$, representing AI-based adaptive assessment, emerges as the most preferable educational technology within the specified numerical decision-making scenario. These findings illustrate the capacity of the proposed framework to integrate complex uncertain assessments and establish a systematic ranking of alternatives under the adopted evaluation conditions.

5.1 Limitations and Future Work

Several limitations should be considered when interpreting the findings of this investigation. First, the numerical application involves only three alternatives and three evaluation criteria and is intended to illustrate the computational implementation of the proposed methodology rather than analyse an empirical educational dataset. Consequently, the reported outcomes demonstrate the computational applicability of the BCPNS-based Hamacher aggregation framework but do not establish its empirical superiority over existing decision-making methodologies. Second, the BCPNS assessments and criterion weights employed in the numerical example are not derived from actual evaluations conducted by educational experts or institutional decision-makers. Third, the proposed methodology has not yet been validated using large-scale benchmark datasets or educational decision-making problems involving multiple organisations.

Future investigations could address these limitations by incorporating assessments from larger groups of educational experts and decision-makers and extending the proposed framework to educational technology selection problems involving more alternatives and evaluation criteria. Additional empirical validation could be undertaken using established decision-making datasets to examine the performance and applicability of the developed aggregation mechanisms under more diverse assessment conditions. Furthermore, systematic comparisons with existing MCDM methodologies and fuzzy, bipolar, complex-valued and neutrosophic aggregation frameworks would provide a broader basis for evaluating the relative capabilities, computational characteristics and practical effectiveness of the proposed BCPNS-based Hamacher aggregation approach.

6. AUTHOR CONTRIBUTIONS

Conceptualization, M.K. and R.V.; methodology, M.K.; software, S.Y. and H.D.; validation, M.K., A.N.; formal analysis, R.V.; investigation, N.K.; resources, H.D.; data curation, S.Y.; writing—original draft preparation, M.K.; writing—review and editing, H.D. and R.V.; visualization, A.N.; supervision, N.K. All authors have read and agreed to the published version of the manuscript."

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8. DATA AVAILABILITY STATEMENT

No datasets were generated or analyzed during the current study. The numerical data used in the illustrative decision-making example are provided within the manuscript.

9. CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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