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Inverse Data Envelopment Analysis Models for Inputs/Outputs Estimation in Two-Stage Processes

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ABSTRACT

Considering the interior of decision-making units (DMUs) is essential when evaluating a system's performance in the practical and real-world circumstances. Knowing what happens inside a DMU allows a more accurate study of relevant process. It identifies the efficiency, and inefficiency of the sub-units in the system being evaluated. This study focuses on two-stage inverse data envelopment analysis (DEA) problems. In these problems, a portion of the outputs from the first period is used as inputs in the second stage. For this purpose, several models are offered to address the input/output estimation problem, in which decision makers should deal with intermediate products and shared and non-shared inputs in a two-stage system. Furthermore, the proposed models are examined using a window DEA because of the importance of assessing repetitive processes in some two-stage systems. Next, an Iranian bank is considered as a case study to further elucidate using the presented models. Finally, we present a conclusion and suggestions for further research.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric method to evaluate the performance of a number of decision-making units (DMUs). In DEA problems, multiple inputs of a DMU are consumed to generate multiple outputs. It is assumed that each DMU contains at least one non-zero input and output. Charnes *et al.,* [1] were the first to introduce DEA in 1974. Furthermore, inverse DEA is a critical, and commonly used subfield of DEA in economics, management, and real-world applications. Wei *et al.,* [2] discussed this subject by asking the following question: if the inputs of under-evaluation DMU (DMU_o) increase, how much its outputs will increase such that the efficiency of that DMU remains unchanged. The issue described is referred to as the output estimation problem. Hadi-Vencheh et al. [3] subsequently addressed the same question about the input estimation problem. The mentioned problem deals with the estimation of the inputs as the outputs of the DMU_o increase,

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and the efficiency score remained at the same level. Furthermore, the output/input estimation problem by increasing the inputs/outputs and improving the performance of $\mathrm{DMU_o}$ was raised [4].

The classical DEA comprises one-step processes, where the DMU's structure is limited to inputs and outputs. Gradually, these simple one-step DMUs replaced by DMUs with multiple stages as network processes became more complex. Therefore, it is crucial to handle these intricate systems, which are widely applicable in the actual world and need appropriate instruments. Analyzing the subprocess and the DMU's overall performance is made easier with an understanding of the structure of the DMU. Neglecting the intermediate structure of DMUs can hinder the accurate analysis of complex real-world processes, including but not limited to the reallocation of royalties in the technological innovations, investments, and allocating technological resources at different stages. Furthermore, classical models fail to accurately assess performance, particularly in cases where the proportion of intermediate output allocated is susceptible. Besides, comprehending how to allocate intermediate productions enhances the system's proficiency and intelligence, enabling decision-makers to make more realistic decisions from a strategic point of view. Jian Feng [5] proposed a two-stage DEA model that simultaneously takes into account the intermediate products and input structure in order to address the problems with classical DEA. The model's goal was to evaluate efficiency by using the intermediate products from the first stage as inputs in the second stage. The model they proposed considered the input structure of DMUs based on the distinction among the inputs assigned to each stage and the common ones.

As mentioned before, the examination of the internal structure of DMU holds significant importance in the practical and real-world cases. Managers and decision-makers require strong theoretical and practical support to manage overall performance, and effectively identify the causes of inefficiencies. In addition, inverse DEA is important in theoretical and real-world settings. Theoretically, managers may use credible scientific data to support their decisions about resource allocation and product delivery. In practice, they can optimally allocate resources and adjust product supply for a set of DMUs while keeping predefined conditions. However, in the classic inverse DEA models provided in the literature (e.g., [2], [3], and [4]), the internal structure of DMUs was ignored, and DMUs were considered black boxes. Consequently, the analyses provided for the input/output estimation problem were made independent of the DMU’s interior. Then, issues related to efficiency, inefficiency, causes of inefficiency, and actions that can be taken to improve or eradicate inefficiency will not be in line with reality in terms of various factors, such as common inputs, intermediate outputs, intermediate inputs, and other elements that have an impact on the evaluation system, but were not considered in the previous models.

This paper considers the inverse DEA as a suitable approach to estimate inputs/outputs, and resource management in two-stage processes to address the shortcomings mentioned. In this paper, we present the inverse case of the Jian Feng [5] model, which permits the evaluation and analysis of system performance with intermediate products and input structures taken into account simultaneously. Moreover, the concerns mentioned above are examined from point of view of window DEA. Our methods can help decision-makers and managers make their policies based on what happens in the system under evaluation.

The most important novelties of this paper are as follows:

i. This study presents the inverse case of two-stage DEA models, which address both the
ii. intermediate output, and the input structure while aiming to estimate all or a portion of
iii. inputs and improve system performance.
iv. The expression and proof of necessary and sufficient conditions for estimating the output of DMU in the two-stage models are described.

v. Considering the importance of analyzing repetitive processes in some two-stage systems, the problem is discussed from the viewpoint of window DEA. Consequently, models, applications, and some properties of mentioned system are discussed.

The remaining parts of the paper are organized as follows: Section 2 contains the literature review. Section 3 provides a background of inverse DEA, multi-objective programming (MOP), and two-stage DEA models. In section 4, we provide modeling of the inverse case of two-stage DEA and present some properties of the problem. Then, a numerical example is given to examine the models in section 5. Finally, the conclusion is given in Section 6.

2. Literature review

DEA models asses the performance of a number of DMUs, which utilizes multiple inputs to generate multiple outputs. Banker *et al.,* [6] categorized the efficiency of a DMU into technical efficiency, and scale efficiency. The study conducted by Färe *et al.,* [7] examined the impact of probability on technical efficiency. In order to determine the total efficiency of the DMU, Kao [8] calculated the regressive mean of the individual outputs' efficiencies. Charnes et al. [9] conducted a pioneering investigation into internal information loss within a single-mode model, employing a two-stage methodology to analyze military recruitment. In classic DEA methods, regardless of the middle phases of production procedure, a DMU is considered a "black box" in which the initial inputs to generate outputs are in a single process, i.e., the inner form of the DMU is often ignored. There is a lack of information regarding the specific component of a DMU that is responsible for inefficiency, as stated by Lewis, Mallikarjun, and Sexton [10], as well as Wang, Huang, Wu, and Liu [11]. Although more information was obtained on the inefficiency of DMUs by decomposing the performance into various components, these studies cannot open the black box in terms of the concentration on the structure of DEA models. Furthermore, based on the internal structure of DMUs, many researchers attempted to break through the "black box."

In recent years, Färe and Grosskopf [12], Chen and Zhu [13], Liang *et al.,* [14], and Chen *et al.,* [15] achieved notable advancements in the expansion of two-stage models within the field of DEA network models. In order to evaluate the efficacy of DMUs with a two-stage internal structure, they proposed a two-stage DEA model. In the initial phase, their model transforms the inputs into intermediate products or middle outputs. In subsequent stage, intermediate products are developed to generate the ultimate products. In two-stage DEA models, DMUs are modeled as two-subunit systems where the intermediate products of first stage are regarded as subunit inputs of next stage. While a two-stage DEA model assists decision-makers in examining a DMU's internal structure and operational processes, it helps identify unsuitable inputs in the subunits [16]; [17]). The two-stage DEA models have the capability to assess the overall performance of the DMU and decompose it into the performance of individual stages through the modelling of inter stages' relationships. Kao and Hwang [18] posit in their two-stage DEA model that the overall efficiency of the operation is equal to the product of the efficiencies of the two stages.

Typical two-stage DEA models focus initially on the interior form of DMUs or the interdependence of various sub-processes on DMUs. Using these models is not suitable or sufficient when inputs or middle values have complex forms. A few researchers investigated this matter and improved the models with respect to the input distribution at various stages. For instance, Yu and Fan [19] and Chen *et al.,* [20] provided a two-stage DEA model with common inputs between the first and second stages. Liang *et al.,* [21] and Chen *et al.,* [22] provided a two-stage model with added inputs to the next stage. Castelli et al., [23] gathered a wide range of categorized models and techniques specifically designed for different multi-stage architectures, beginning with a basic foundation. Fukuyama and Weber [24] developed a slack-based model to address a two-stage system that involves undesirable outputs. Liu *et al.,* [25] examined DEA models for processing network structures. Cook *et al.,* [26] presented a two-stage DEA model for assessing the performance of a two-stage network procedure in which some inputs are straightly related to both stages. They extended their model to a two-stage network where the second stage contains inputs, and parts of the first stage's outputs are not always regarded as inputs of the subsequent stage. Zha and Liang [27] provided a method for handling two-stage process where primary input can be flexibly assigned. Nevertheless, there are several issues with their strategy. It forbids utilizing more inputs directly in a later phase and sharing intermediate products. Yu and Shi [28] improved the approach put forward by Zha and Liang [27], who looked more closely at the two-stage procedure's structure. The suggested approach involves using supplementary inputs and a portion of the outputs from that first stage as inputs in the subsequent stage. Despite providing a parametric approach to examine two-stage structure described, the structure could not accommodate the existence of shared inputs or ultimate outputs directly derived from stage one. A two-stage DEA model was introduced by Yu and Shi [28], wherein additional inputs are incorporated into the second stage, with the part of middle products considered as the final output. Izadikhah *et al.,* [4] proposed a two-stage model where DMUs are composed of two sub-DMUs. The sub-DMU of the first stage uses a fraction of the intermediate products from the sub-DMU of the second stage. The main input may be assigned to one of the two subordinate DMUs. Moreover, the second stage immediately consumes new inputs, enabling sub-DMU in the first stage to generate ultimate outputs.

Inverse DEA was first presented by Zhang and Cui [29] for evaluating efficiency of sub-units in a project evaluation system in China. A typical form of inverse DEA was introduced by Wei et al. [2] to handle the topic as an output estimation problem. To achieve this, they proposed the multi-objective linear programming model (MOLP). They then tackled the issue by transforming the MOLP into a linear programming problem with a single objective function. Many researchers examined and modified this concept following the introduction of inverse DEA. Jahanshahloo *et al.,* [30], [31]) expanded upon the models proposed by Wei *et al.,* [2] and improved upon them by introducing surplus inputs. Hadi-Vencheh *et al.,* [3] presented the input estimation problem and modified the sufficient conditions provided by Wei *et al.,* [2]. In their study, Hadi-Vencheh *et al.,* [3] recommended using a strong efficient solution instead of a weakly efficient solution to estimate the input with increased output. Besides, several relevant scholarly papers have explored the use of inverse DEA, including the existence of time dependence [32], fuzzy DEA [33], and the inverse DEA problem using pricing information [34].

This study investigated input/output estimation problem in a two-stage DEA process. It considers the input, and output factors and the internal structure of the under evaluation DMU. This approach will make it easier for managers, investors, and decision-makers to create an expert system. We provide the inverse case of Jian Feng [5] in order to evaluate and analyze the performance of a two-stage DEA method that considers both intermediate products and input structures. Furthermore, we analyze the mentioned challenges, namely input/output estimation problems in a two-stage process, using a window DEA approach. This approach simultaneously considers the internal processes of DMU, including shared inputs, intermediate inputs, and outputs.

Subsequently, we present basic concepts and models used in our proposed models and methodology.

3. Background

3.1. Multi-Objective Programming

Let $g: \mathbb{R}^p \to \mathbb{R}^q$ and $g': \mathbb{R}^p \to \mathbb{R}^k$ are vector functions, an MOP problem generally is as follows:

$$\begin{aligned} &\text{Minimize} \quad g(x) \\ &\quad \text{Subject to} \\ &\qquad g'_j(x) \le 0 \qquad , \qquad j = 1,\dots,k \end{aligned} \tag{1}$$

in which $g(x) = (g_1(x),\dots,g_q(x))$ and $g'(x) = (g'_1(x),\dots,g'_k(x))$. Furthermore, the set $X = \{x \in \mathbb{R}^p;\ \ g_j(x) \le 0,\ j = 1, 2, \dots, k\}$ is called a set of feasible solutions. Since there is usually no optimal solution for these problems, the efficient (Pareto), and weakly efficient (weak Pareto) solutions are defined as follows:

Definition 1. A feasible solution $x^* \in X$ is called a Pareto solution for the MOP if there is no feasible solution $x^o \in X$ that $g_j(x^o) \le g_j(x^*);\ j = 1,\dots,q$.

Definition 2. A feasible solution $x^* \in X$ is a weak Pareto solution for MOP if there is no feasible solution $x^o \in X$ that $g_j(x^o) < g_j(x^*);\ \ j = 1,\dots,q$.

3.2. Inverse DEA

Suppose that n DMUs with m inputs and s outputs are given and DMU_o; $o \in \{j = 1, 2, \dots, n\}$ is the DMU under evaluation. Besides, the vectors $X_j \in \mathbb{R}^m \geq\neq 0$; $(j = 1, 2, \dots, n)$ and $Y_j \in \mathbb{R}^s \geq\neq 0$; $(j = 1, 2, \dots, n)$ are input and output vectors corresponding to DMU_j, respectively.

The envelopment output-oriented model for efficiency evaluation of DMU_o is as follows:

$$\begin{aligned} &\varphi_o = \text{Maximize } \varphi \\ &\quad \text{Subject to} \\ &\qquad \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} \qquad , \qquad i = 1, 2, \dots, m \\ &\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge \varphi y_{ro} \qquad , \qquad r = 1, 2, \dots, s \\ &\qquad \lambda_j \ge 0 \qquad , \qquad j = 1, 2, \dots, n \end{aligned} \tag{2}$$

Wei *et al.,* [2], in 2000, proposed inverse DEA as a solution to "output estimation problem": how much should the output of the unit rise if the inputs of unit under evaluation DMU_o increase while keeping the efficiency index (φ_o) constant?

The same question was then posed from an alternate point of view by Hadi-Vencheh *et al.,* [3] in 2008 as an "input estimation problem," in which the amount of increase in DMU_o's inputs would be assessed if its outputs increased, and the efficiency score remained unchanged. Estimating an evaluation unit's output while increasing inputs and efficiency was a problem that Jahanshahloo *et al.,* [30] addressed in 2004. Subsequently, they investigated the input estimation problem concerning a steady efficiency index and increasing outputs. In addition, several models for estimating the input surplus were provided [31].

The output estimation problem is defined as follows:

Suppose that certain inputs of DMU_o are increased to a specific amount. How much will the output of this unit increase so that the performance of this new unit remains unchanged with respect to other units?

Assume that inputs of $\mathrm{DMU_o}$ are increased from $\mathrm{X_o}$ to $\alpha_o = \mathrm{X_o} + \Delta\mathrm{X_o}$ so that $\Delta\mathrm{X_o} \geq 0$ and $\Delta\mathrm{X_o} \neq 0$. Then, by Keeping the efficiency index (φ_o) unchanged, how should be estimated the output vector β_o^* in which:

$$\beta_o^* = (\beta_{1o}^*, \beta_{2o}^*, \ldots, \beta_{so}^*)^t = \mathrm{Y_o} + \Delta\mathrm{Y_o}\,; \qquad \Delta\mathrm{Y_o} \geq 0$$

To answer aforementioned question, the following MOLP was proposed by Wei *et al.,* [2]:

$$\begin{aligned} &\text{Maximize } (\beta_{1o}, \beta_{2o}, \ldots, \beta_{so}) \\ &\quad \text{subject to} \\ &\quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \alpha_{io}, && i = 1, 2, \ldots, m \\ &\quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi_o^* \beta_{ro}, && r = 1, 2, \ldots, s \\ &\quad \beta_{ro} \geq y_{ro}, && r = 1, 2, \ldots, s \\ &\quad \lambda_j \geq 0, && j = 1, 2, \ldots, n \end{aligned} \tag{3}$$

In model (3), $(\lambda, \beta_o) \in \mathbb{R}^p \times \mathbb{R}^s$ is the vector of variables, and φ_o^* is the optimal solution of model (2). Wei *et al.,* [2] studied the output estimation problem using the following theorem:

Theorem 1. Let inputs of $\mathrm{DMU_o}$ are increased from $\mathrm{X_o}$ to $\alpha_o = \mathrm{X_o} + \Delta\mathrm{X_o}$ such that $\Delta\mathrm{X_o} \geq 0$ and $\Delta\mathrm{X_o} \neq 0$:

a) If $\varphi_o^* = 1$, then $\varphi(\alpha_o, \varphi_o^*\mathrm{Y_o}) = \varphi(\mathrm{X_o}, \mathrm{Y_o})$.

b) Suppose that $\varphi_o^* > 1$ i.e., $\mathrm{DMU_o}$ is inefficient. If (λ^*, β_o^*) is a weakly Pareto efficient for the MOLP

(3) and outputs of $\mathrm{DMU_o}$ are increased from $\mathrm{Y_o}$ to β_o^*, Then the performance level of new DMU will remain unchanged. In other words:

$$\varphi(\alpha_o, \beta_o^*) = \varphi(\mathrm{X_o}, \mathrm{Y_o})$$

Conversely, if (λ^*, β_o^*) is a feasible solution for the model (3), and efficiency index of the new DMU remains unchanged, i.e., $\varphi(\alpha_o, \beta_o^*) = \varphi(\mathrm{X_o}, \mathrm{Y_o})$, then (λ^*, β_o^*) is a weak Pareto solution for this problem.

In 2004, Jahanshahloo *et al.,* [30] developed the output estimation model provided by Wei *et al.,* [2] considering the condition that the efficiency of new unit is improved from φ_o^* by amount of $\%\,\eta$ where $0 \leq \eta \leq \frac{100\,(\varphi_o^*-1)}{\varphi_o^*}$. They also defined the concept of inputs surplus as follows [31]:

Definition 3. The inputs surplus of a DMU is the maximum input reduction to maintain the unit's output and performance level.

The following model was presented to determine the amount of input surplus, considering the efficiency improvement in the output estimation problem [31]:

$$\begin{aligned} &\text{Maximize } \delta_o = (\delta_{1o}, \delta_{2o}, \ldots, \delta_{mo}) \\ &\quad \text{Subject to} \\ &\quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq (\alpha_{io} - \delta_{io}), && i = 1, 2, \ldots, m \\ &\quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq \left(1 - \tfrac{\eta}{100}\right) \varphi_o^* \beta_{ro}, && r = 1, 2, \ldots, s \\ &\quad \sum_{j=1}^{n} \lambda_j = 1 \end{aligned} \tag{4}$$

$$\alpha_{io} \geq 0 \quad , \quad i = 1, 2, \ldots, m$$
$$\lambda_j \geq 0 \quad , \quad j = 1, 2, \ldots, n$$

In 2008, Hadi-Vencheh *et al.* [3] considered and answered the following question about an input estimation problem:

If specified outputs of DMU_o are increased to a given amount, then how would the inputs of DMU_o increase such that the performance score of this new unit remains unchanged with respect to the other units? It means suppose that outputs of DMU_o increased from Y_o to $\beta_o = Y_o + \Delta Y_o$, ($\Delta Y_o \geq 0$ and $\Delta Y_o \neq 0$), then how should the inputs vector α_o^* be estimated such that the performance level stays at θ_o in which $\alpha_o^* = (\alpha_{1o}^*, \ldots, \alpha_{mo}^*)^t = X_o + \Delta X_o$, $\Delta X_o \geq 0$.

To answer the above question, the following MOLP was proposed:

$$\text{Minimize } \alpha_o = (\alpha_{1o}, \alpha_{2o}, \ldots, \alpha_{mo}) \tag{5}$$
Subject to
$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_o^* \alpha_{io} \quad , \quad i = 1, 2, \ldots, m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \beta_{ro} \quad , \quad r = 1, 2, \ldots, s$$
$$\alpha_{io} \geq 0 \quad , \quad i = 1, 2, \ldots, m$$
$$\lambda_j \geq 0 \quad , \quad j = 1, 2, \ldots, n$$

In the model (5), (α_o, λ) is the vector of variables, and θ_o^* is the optimal value of the input-oriented model for efficiency evaluation of DMU_o. To address this model, the following theorems were provided by Hadi-Vencheh *et al.,* [3].

Theorem 2. If (α_o^*, λ^*) is a Pareto efficient solution for the MOLP (5), then:

$$\theta(\alpha_o^*, , \beta_o) = \theta(X_o, Y_o)$$

Theorem 3. If (α_o^*, λ^*) is a weakly Pareto efficient solution for the MOLP (5) in which $\alpha_o^* > X_o$, then:

$$\theta(\alpha_o^*, \beta_o) = \theta(X_o, Y_o)$$

3.3. Two-stage DEA

In the classic DEA models, DMU is a "black box" (represented by the dotted line in Figure 1), in which either the information is missing or its internal structure is not considered. Consequently, the DMU in these models exclusively utilizes primary inputs to generate final outputs, irrespective of the different stages of its internal processes. Therefore, it is impossible to measure the performance of sub-processes or the inefficiency of DMUs to enhance the performance of internal operations and the total efficiency of DMUs. The significant features of the model presented by Jian Feng [5] are as follows:

i. Modeling the interior form of a DMU is within a network structure.
ii. Inputs of the DMU are arranged into three groups: primary inputs, added inputs, and common inputs.
iii. The model is capable of reviewing the performance values based on the traditional model and can evaluate it using the conventional two-stage model.

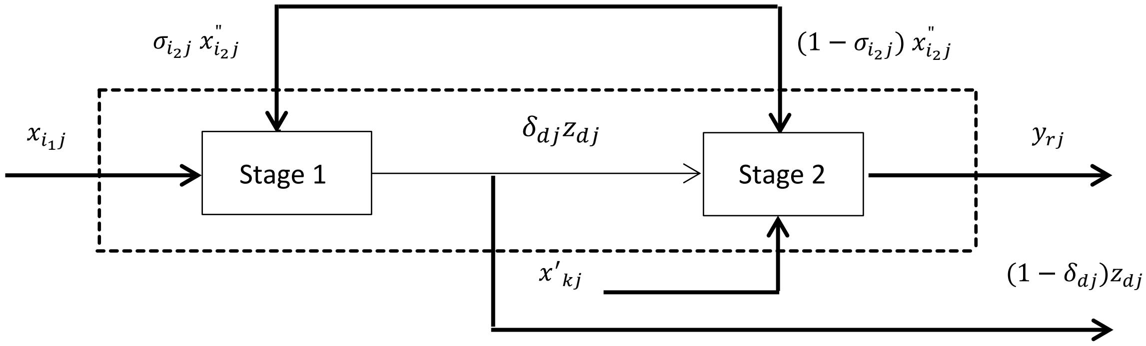

Fig. 1. A Two-stage Network with common inputs and free middle productions

Consider a group of DMUs like $\mathrm{DMU_j}$; $(j = 1, \ldots, n)$. Assume that each of these DMUs takes part in two-stage DEA procedure shown in Figure 1 so that for each DMU, there are three types of inputs: the primary, the additional, and the shared inputs. The primary inputs are shown as $x_{i_1j};\ i_1 \in I_1$, the shared inputs are as $x''_{i_2j};\ i_2 \in I_2$ in which $I_1 \cup I_2 = \{1, \ldots, m\}$ & $I_1 \cap I_2 = \emptyset$, σ_{i_2j} and $(1-\sigma_{i_2j})$ are the amount of input x''_{i_2j} that are assigned to the stage 1 and stage 2, respectively. Similar to the constraints mentioned in Cook and Hababou [35], $L^1_{i_2j} \le \sigma_{i_2j} \le L^2_{i_2j};\quad (i_2 \in I_2,\ L^1_{i_2j} > 0,\ L^2_{i_2j} < 1;\ j = 1, \ldots, n)$.

It is obvious that the closer the value of σ_{i_2j} is to zero, it is expected that the efficiency of stage 1 to be higher and the efficiency of the stage 2 to be lower. Conversely, the closer σ_{i_2j} is to 1, the higher the efficiency of the stage 2 and the lower the efficiency of the stage 1.

Furthermore, the additional inputs are stated as $x'_{kj};\ (k = 1, \ldots, q)$. $z_{dj};\ (d = 1, \ldots, D)$, is the output of the first stage that part of it i.e., $\delta_{d_j} z_{d_j}$ forms input of the second stage. Other part of it i.e., $(1 - \delta_{d_j} z_{d_j})$ is considered as the final output of DMU in which $H^1_{dj} \le \delta_{d_j} \le H^2_{dj};\ H^1_{dj} > 0$ and $H^2_{dj} \le 1$, and final outputs of the second stage are represented as $y_{rj};\ (r = 1, \ldots, S)$.

Now, consider two-stage procedure shown in Figure 1 with $(H^2_{dj} < 1)$. More precisely, for each DMU, two decisions must be made, including the allocation of resources and the reassignment of intermediate products. Considering a two-stage procedure with common inputs and free middle products, the performance score of the $\mathrm{DMU_o}$ in the first and second stage is denoted by Φ^{SF1}_o and Φ^{SF2}_o, respectively. Furthermore, overall efficiency Φ^{SF}_o is calculated as follows (Jian Feng [5]):

$$\Phi^{SF1}_o = \frac{\sum_{d=1}^{D} u'_d\, z_{do}}{\sum_{i_1 \in I_1} v_{i_1}\, x_{i_1o} + \sum_{i_2 \in I_2} v''_{i_2}\, \sigma_{i_2o}\, x''_{i_2o}}$$

$$\Phi^{SF2}_o = \frac{\sum_{r=1}^{S} u_r\, y_{ro}}{\sum_{i_2 \in I_2} v''_{i_2}\, (1-\sigma_{i_2o})\, x''_{i_2o} + \sum_{k=1}^{q} v'_k\, x'_{ko} + \sum_{d=1}^{D} u'_d\, \delta_{do}\, z_{do}}$$

Moreover, the total performance for $\mathrm{DMU_o}$ is considered as $w_1\, \Phi^{SF1}_o + w_2\, \Phi^{SF2}_o = \Phi^{SF}_o$, where:

$$w_1 = \frac{\sum_{i_1 \in I_1} v_{i_1} x_{i_1 o} + \sum_{i_2 \in I_2} v''_{i_2} \sigma_{i_2 o} x''_{i_2 o}}{\sum_{i_1 \in I_1} v_{i_1} x_{i_1 o} + \sum_{i_2 \in I_2} v''_{i_2} x''_{i_2 o} + \sum_{k=1}^{q} v'_k x'_{ko} + \sum_{d=1}^{D} u'_d \delta_{do} z_{do}}$$

and

$$w_2 = \frac{\sum_{i_2 \in I_2} v''_{i_2} (1 - \sigma_{i_2 o}) x''_{i_2 o} + \sum_{k=1}^{q} v'_k x'_{ko} + \sum_{d=1}^{D} u'_d \delta_{do} z_{do}}{\sum_{i_1 \in I_1} v_{i_1} x_{i_1 o} + \sum_{i_2 \in I_2} v''_{i_2} x''_{i_2 o} + \sum_{k=1}^{q} v'_k x'_{ko} + \sum_{d=1}^{D} u'_d \delta_{do} z_{do}}$$

DEA model to assess the total efficiency of the DMU_o is presented as the following fractional model:

$$\Phi_o^{SF} = \text{Maximize} \frac{\sum_{d=1}^{D} u'_d z_{do} + \sum_{r=1}^{S} u_r y_{ro}}{\sum_{i_1 \in I_1} v_{i_1} x_{i_1 o} + \sum_{i_2 \in I_2} v''_{i_2} x''_{i_2 o} + \sum_{k=1}^{q} v'_k x'_{ko} + \sum_{d=1}^{D} u'_d \delta_{do} z_{do}}$$

$$\text{subject to} \qquad (6)$$

$$\frac{\sum_{d=1}^{D} u'_d z_{dj}}{\sum_{i_1 \in I_1} v_{i_1} x_{i_1 j} + \sum_{i_2 \in I_2} v''_{i_2} \sigma_{i_2 j} x''_{i_2 j}} \le 1, \qquad j = 1, \dots, n$$

$$\frac{\sum_{r=1}^{S} u_r y_{rj}}{\sum_{i_2 \in I_2} v''_{i_2} (1 - \sigma_{i_2 j}) x''_{i_2 j} + \sum_{k=1}^{q} v'_k x'_{kj} + \sum_{d=1}^{D} u'_d \delta_{dj} z_{dj}} \le 1, \qquad r = 1, \dots, S$$

$$L^1_{i_2 j} \le \sigma_{i_2 j} \le L^2_{i_2 j}, \qquad i_2 \in I_2; \; j = 1, \dots, n$$

$$H^1_{dj} \le \delta_{dj} \le H^2_{dj}, \qquad d = 1, \dots, D; \; j = 1, \dots, n$$

$$v_{i_1} \ge 0, \; v''_{i_2} \ge 0, \; v'_k \ge 0, \qquad i_1 \in I_1; \; i_2 \in I_2; \; k = 1, \dots, q$$

Remark 1. Since the constraints $H^1_{dj} \le \delta_{dj} \le H^2_{dj}$ and $L^1_{i_2 j} \le \sigma_{i_2 j} \le L^2_{i_2 j}$ included in the model (6) are in terms of managerial preferences, then they can be considered definite amounts. Therefore, if these assumptions make the model infeasible, these can be modified by using the goal programming method.

Finally, using Charnes and Cooper Transformations [36], model (7) is obtained as below:

$$\Phi_o^{SF} \quad \text{Maximize} \sum_{d=1}^{D} u'_d z_{do} + \sum_{r=1}^{S} u_r y_{ro}$$

$$\text{subject to} \qquad (7)$$

$$\sum_{d=1}^{D} u'_d z_{dj} - \sum_{i_1 \in I_1} v_{i_1} x_{i_1 j} - \sum_{i_2 \in I_2} v''_{i_2} \sigma_{i_2 j} x''_{i_2 j} \leq 0 \qquad , \qquad j = 1, \dots, n$$

$$\sum_{r=1}^{S} u_r y_{rj} - \sum_{i_2 \in I_2} v''_{i_2} (1 - \sigma_{i_2 j}) x''_{i_2 j} + \sum_{k=1}^{q} v'_k x'_{kj} + \sum_{d=1}^{D} \eta_d z_{dj} \leq 0 \qquad j = 1, \dots, n$$

$$\sum_{i_1 \in I_1} v_{i_1} x_{i_1 o} + \sum_{i_2 \in I_2} v''_{i_2} x''_{i_2 o} + \sum_{k=1}^{q} v'_k x'_{ko} + \sum_{d=1}^{D} \eta_d z_{do} = 1 \qquad r = 1, \dots, S$$

$$u_r \geq 0, \; u'_d, \eta_d \geq 0 \qquad , \qquad r = 1, \dots, S; \; d = 1, \dots, D$$

$$v_{i_1} \geq 0, \; v''_{i_2} \geq 0, \; v'_k \geq 0 \qquad , \qquad i_1 \in I_1; \; i_2 \in I_2; \; k = 1, \dots, q$$

in which $v_{i_1} = tv_{i_1}$, $u_r = tu_r$, $u'_d = tu'_d$, $v'_k = tv'_k$, $\eta_d = u'_d \delta_{dj}$. Model (7) expresses the total performance of DMU_o for the internal process shown in Figure 1. This model considers the inputs structure and middle productions at the same time. Jian Feng [5] showed that if $u_r \geq \varepsilon$, $\eta_d, u'_d \geq \varepsilon$, $v_{i_1} \geq \varepsilon$, $v''_{i_2} \geq \varepsilon$, $v'_k \geq \varepsilon$, $(r = 1, \dots, S;\; i_1 \in I_1;\; i_2 \in I_2;\; d = 1, \dots, D;\; k = 1, \dots, q)$ in the model (7), then the model has a non-zero optimal solution that satisfies the constraints η_d, $u'_d \geq 0$, $u_r \geq 0$, $v_{i_1} \geq 0$, $v''_{i_2} \geq 0$, $v'_k \geq 0$, $(d = 1, \dots, D;\; r = 1, \dots, S;\; i_1 \in I_1;\; i_2 \in I_2;\; k = 1, \dots, q)$.

4. Modelling

The interior structure of DMUs plays a crucial role in most practical and real difficulties while studying two-stage processes. Decision-makers require theoretical and practical support to optimize the overall efficiency of units, and thoroughly study the underlying causes of inefficiency. As a result, this section presents the inverse version model that Jian and Feng [5] suggested. This model helps decision makers analyze the performance assessment of DMUs by taking into account input structures and intermediate products at the same time. Furthermore, this study aims to articulate the essential and sufficient conditions to assess the outputs and all or a portion of the inputs involved in improving performance. Furthermore, since some production processes are taken place and repeated in several periods, the models offered are also expressed from a window DEA perspective.

We consider the dual form of the model (7) as follows:

$$\theta_o = \text{Minimize } \theta$$

$$\text{subject to} \qquad (8)$$

$$\sum_{j=1}^{n} \rho_j z_{dj} \leq \theta z_{do} \qquad , \qquad d = 1, 2, \dots, D$$

$$\sum_{j=1}^{n} \lambda_j x_{i_1 j} \leq \theta x_{i_1 o} \qquad , \qquad i_1 \in I_1$$

$$\sum_{j=1}^{n} \rho_j x'_{kj} \leq \theta x'_{ko} \qquad , \qquad k = 1, 2, \dots, q$$

$$\sum_{j=1}^{n} ((1 - \sigma_{i_2 j}) \rho_j + \sigma_{i_2 j} \lambda_j) x''_{i_2 j} \leq \theta x''_{i_2 o} \qquad , \qquad i_2 \in I_2$$

$$\sum_{j=1}^{n} \lambda_j z_{dj} \geq z_{do} \qquad , \qquad d = 1, 2, \dots, D$$

$$\sum_{j=1}^{n} \rho_j y_{rj} \ge y_{ro} \quad , \quad r = 1, 2, \ldots, S$$
$$\rho_j, \lambda_j \ge 0 \quad , \quad j = 1, 2, \ldots, n$$

The output-oriented of the model (8) is:

$$
\begin{aligned}
\varphi_o \; &= \text{Maximize } \varphi \\
&\text{subject to} \qquad\qquad (9) \\
&\sum_{j=1}^{n} \rho_j z_{dj} \le z_{do} \quad , \quad d = 1, 2, \ldots, D \\
&\sum_{j=1}^{n} \lambda_j x_{i_1 j} \le x_{i_1 o} \quad , \quad i_1 \in I_1 \\
&\sum_{j=1}^{n} \rho_j x'_{kj} \le x'_{ko} \quad , \quad k = 1, 2, \ldots, q \\
&\sum_{j=1}^{n} ((1 - \sigma_{i_2 j})\rho_j + \sigma_{i_2 j}\lambda_j) x''_{i_2 j} \le x''_{i_2 o} \quad , \quad i_2 \in I_2 \\
&\sum_{j=1}^{n} \lambda_j z_{dj} \ge \varphi z_{do} \quad , \quad d = 1, 2, \ldots, D \\
&\sum_{j=1}^{n} \rho_j y_{rj} \ge \varphi y_{ro} \quad , \quad r = 1, 2, \ldots, S \\
&\rho_j, \lambda_j \ge 0 \quad , \quad j = 1, 2, \ldots, n
\end{aligned}
$$

Remark 2. The efficiency obtained from two models (8) and (9) is the total efficiency. Therefore, multiplier models should be used to obtain component efficiency.

Remark 3.

a) It is easy to show that $\theta_o^* \le 1$ and $\varphi_o^* \ge 1$.

b) If $\theta_o^* = 1$, then DMU_o is at least weakly efficient. The same result is held for DMU_o in output orientation when $\varphi_o^* = 1$.

4.1. Output Estimation

Now, to apply and extend the output estimation problem provided by Wei *et al.,* [2] for the two-stage DEA framework considering the condition shown in Figure 1, and improving efficiency, suppose that inputs of DMU_o changes from (X_o, X''_o, X'_o) to $(\alpha_o, \mu_o, \gamma_o)$ in such a way that:

$$(\alpha_o, \mu_o, \gamma_o) = (\alpha_{i_1 o}, i_1 \in I_1; \; \mu_{i_2 o}, i_2 \in I_2; \; \gamma_{ko}, k = 1, \ldots, q) = (X_o, X''_o, X'_o) + (\Delta X_o, \Delta X''_o, \Delta X'_o)$$

where $(\Delta X_o, \Delta X''_o, \Delta X'_o) \ge 0$ and $(\Delta X_o, \Delta X''_o, \Delta X'_o) \ne 0$.

Now, output vector β_o that $\beta_o = Y_o + \Delta Y_o$ and $\Delta Y_o \ge 0$ is estimated such that the efficiency of DMU_o is improved by $(1 - \frac{\eta}{100})\varphi_o^*$. To estimate the output vector, the following MOLP is considered:

$$
\begin{aligned}
&\text{Maximize } (\beta_{ro};\quad r = 1, \dots, S) \\
&\quad \text{subject to} && && (10) \\
&\quad \sum_{j=1}^{n} \rho_j z_{dj} \le z_{do} && , && d = 1, 2, \dots, D \\
&\quad \sum_{j=1}^{n} \lambda_j x_{i_1 j} \le \alpha_{i_1 o} && , && i_1 \in I_1 \\
&\quad \sum_{j=1}^{n} \rho_j x'_{kj} \le \gamma_{ko} && , && k = 1, 2, \dots, q \\
&\quad \sum_{j=1}^{n} ((1 - \sigma_{i_2 j})\rho_j + \sigma_{i_2 j}\lambda_j) x''_{i_2 j} \le \mu_{i_2 o} && , && i_2 \in I_2 \\
&\quad \sum_{j=1}^{n} \lambda_j z_{dj} \ge (1 - \frac{\eta}{100})\varphi_o^* z_{do} && , && d = 1, 2, \dots, D \\
&\quad \sum_{j=1}^{n} \rho_j y_{rj} \ge (1 - \frac{\eta}{100})\varphi_o^* \beta_{ro} && , && r = 1, 2, \dots, S \\
&\quad \rho_j , \lambda_j \ge 0,\ \ \beta_{ro} \ge 0 && , && j = 1, 2, \dots, n;\ r = 1, 2, \dots, S
\end{aligned}
$$

Note that in the model (10), $0 < \sigma_{i_2 j} < 1$; $(j = 1, \dots, n;\ \ i_2 \in I_2)$ and $0 \le \eta \le \frac{100(\varphi_o^* - 1)}{\varphi_o^*}$. It is easy to see that if $\eta = 0$, the performance level of unit under evaluation remains unchanged with respect to the other units. Besides, in the case that $\eta = \frac{100(\varphi_o^* - 1)}{\varphi_o^*}$, the unit will be efficient. Moreover, the vector of variables is $(\lambda_j,\ \rho_j,\ \beta_{ro})$; $(r = 1, \dots, S;\ j = 1, \dots, n)$. Furthermore, $(\alpha_{i_1 o},\ \mu_{i_2 o},\ \gamma_{ko})$; $(i_1 \in I_1;\ \ i_2 \in I_2;\ \ k = 1, \dots, q)$ and $(1 - \frac{\eta}{100})\varphi_o^*$ as the optimal solution of model (9) improved by η%, are fixed values. Now, suppose that the newly generated unit after the change in its input and output levels is shown as DMU_{n+1}. To measure the performance of DMU_{n+1}, the following model, derived from the output-oriented model (9), is presented:

$$
\begin{aligned}
&\varphi_{n+1} = \text{Maximize } \varphi \\
&\quad \text{subject to} && && (11) \\
&\quad \sum_{j=1}^{n} \rho_j z_{dj} + \rho_{n+1} z_{do} \le z_{do} && , && d = 1, 2, \dots, D \\
&\quad \sum_{j=1}^{n} \lambda_j x_{i_1 j} + \lambda_{n+1} \alpha_{i_1 o} \le \alpha_{i_1 o} && , && i_1 \in I_1 \\
&\quad \sum_{j=1}^{n} \rho_j x'_{kj} + \rho_{n+1} \gamma_{ko} \le \gamma_{ko} && , && k = 1, 2, \dots, q \\
&\quad \sum_{j=1}^{n} \Big((1 - \sigma_{i_2 j})\rho_j + \sigma_{i_2 j}\lambda_j\Big) x''_{i_2 j} + \Big((1 - \sigma_{i_2 (n+1)})\rho_{n+1} + \sigma_{i_2 (n+1)}\lambda_{n+1}\Big) \mu_{i_2 o} \le \mu_{i_2 o} && && i_2 \in I_2 \\
&\quad \sum_{j=1}^{n} \lambda_j z_{dj} + \lambda_{n+1} z_{do} \ge \varphi z_{do} && , && d = 1, 2, \dots, D \\
&\quad \sum_{j=1}^{n} \rho_j y_{rj} + \rho_{n+1} \beta_{ro} \ge \varphi \beta_{ro} && , && r = 1, 2, \dots, S \\
&\quad \rho_j,\ \lambda_j \ge 0 && , && j = 1, \dots, n+1
\end{aligned}
$$

Now, if the following condition is held for the optimal solution of models (9) and (11), then the efficiency score will be improved:

$$\varphi(\alpha_o, \mu_o, \gamma_o, \beta_o) \leq \varphi(X_o, X''_o, X'_o, Y_o)$$

The model below is used to determine the input surplus:

$$\begin{aligned}
&\text{Maximize } (\delta_{i_1 o}, i_1 \in I_1; \;\; \nu_{i_2 o}, i_2 \in I_2; \;\; \tau_{ko}, k = 1, \ldots, q) \\
&\quad \text{suject to} && && (12) \\
&\sum_{j=1}^{n} \rho_j z_{dj} \leq z_{do} &&, && d = 1, 2, \ldots, D \\
&\sum_{j=1}^{n} \lambda_j x_{i_1 j} \leq (\alpha_{i_1 o} - \delta_{i_1 o}) &&, && i_1 \in I_1 \\
&\sum_{j=1}^{n} \rho_j x'_{kj} \leq (\gamma_{ko} - \tau_{ko}) &&, && k = 1, 2, \ldots, q \\
&\sum_{j=1}^{n} ((1 - \sigma_{i_2 j})\rho_j + \sigma_{i_2 j}\lambda_j) x''_{i_2 j} \leq (\mu_{i_2 o} - \nu_{i_2 o}) &&, && i_2 \in I_2 \\
&\sum_{j=1}^{n} \lambda_j z_{dj} \geq (1 - \tfrac{\eta}{100}) \varphi_o^* z_{do} &&, && d = 1, 2, \ldots, D \\
&\sum_{j=1}^{n} \rho_j y_{rj} \geq (1 - \tfrac{\eta}{100}) \varphi_o^* \beta_{ro} &&, && r = 1, 2, \ldots, S \\
&\rho_j, \lambda_j \geq 0, &&, && j = 1, 2, \ldots, n \\
&\delta_{i_1 o} \geq 0, \; \nu_{i_2 o} \geq 0, \; \tau_{ko} \geq 0 &&, && k = 1, 2, \ldots, q; \; i_1 \in I_1; i_2 \in I_2
\end{aligned}$$

Note that if $(\delta_o^*, v_o^*, \tau_o^*, \lambda^*, \rho^*)$ is a Pareto solution for (12), then the amount of input surplus for the i_1^{th} input is obtained from $\alpha_{i_1 o} - \delta_{i_1 o}^*$.

Regarding the output estimation problem with the condition of efficiency improvement, the following theorems are stated and proved:

Theorem 4. If $(\lambda^*, \beta_o^*, \rho^*)$ is a Pareto solution of model (10) and the output of DMU_o increase from Y_o to β_o^*, then the efficiency score of new DMU will be improved compared with DMU_o. It means that:

$$\varphi(\alpha_o, \mu_o, \gamma_o, \beta_o^*) \leq \varphi(X_o, X''_o, X'_o, Y_o)$$

Proof: Suppose that $(\lambda^*, \beta_o^*, \rho^*)$ is a Pareto solution and $(1 - \frac{\eta}{100})\varphi_o^*$ is the objective value of the model (10). By putting this solution in the model, the following relations are yielded:

$$\sum_{j=1}^{n} \lambda_j^* x_{i_1 j} \leq \alpha_{i_1 o} \quad , \quad i_1 \in I_1$$

and

$$\sum_{j=1}^{n} \rho_j^* y_{rj} \geq (1-\frac{\eta}{100})\varphi_o^* \beta_{ro}^* \quad , \quad r = 1, 2, \ldots, S$$

Since $(1-\frac{\eta}{100})\varphi_o^* \geq 1$, we have:

$$\sum_{j=1}^{n} \rho_j^* y_{rj} \geq (1-\frac{\eta}{100})\varphi_o^* \beta_{ro}^* \geq \beta_{ro}^* \quad , \quad r = 1, 2, \ldots, S$$

So, the following inequality is satisfied:

$$\sum_{j=1}^{n} \rho_j^* y_{rj} \geq \beta_{ro}^* \quad , \quad r = 1, 2, \ldots, S$$

We already know $x_{i_1 o} \leq \alpha_{i_1 o}, \beta_{ro}^* \geq y_{ro}$; $(i_1 \in I_1$ & $r = 1, 2, \ldots, S)$
Now, we define:

$$\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n+1}) = \begin{cases} \bar{\lambda}_j = \lambda_j^* & j \neq n+1 \\ \bar{\lambda}_{n+1} = 0 & j = n+1 \end{cases}$$

and

$$\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_{n+1}) = \begin{cases} \bar{\rho}_j = \rho_j^* & j \neq n+1 \\ \bar{\rho}_{n+1} = 0 & j = n+1 \end{cases}$$

Hence, $(\bar{\lambda}, \bar{\rho})$ and the objective value$(1-\frac{\eta}{100})\varphi_o^*$ are the feasible solution for the model (11), then:

$$\varphi_{(n+1)o}^* \leq (1-\frac{\eta}{100})\varphi_o^* \leq \varphi_o^*$$

It means that $\varphi(\alpha_o, \mu_o, \gamma_o, \beta_o^*) \leq \varphi(X_o, X_o^{"}, X_o', Y_o)$. So, the statement is proved. □

Theorem 5. If $(\lambda^*, \beta_o^*, \rho^*)$ is a Pareto solution for the model (10), then the constraints below are binding i.e.:

$$\sum_{j=1}^{n} \rho_j^* y_{rj} = (1-\frac{\eta}{100})\varphi_o^* \beta_{ro}^* \quad , \quad r = 1, 2, \ldots, S$$

Proof: Based on the absurdum, we assume that there is an index (l) for which the last constraint is not binding, i.e.:

$$\sum_{j=1}^{n} \rho_j^* y_{lj} > (1-\frac{\eta}{100})\varphi_o^* \beta_{lo}^*$$

Now we consider positive and non-zero number (q), such that satisfies the following inequality:

$$0 < q \leq \frac{\sum_{j=1}^{n} \rho_j^* y_{lj} - (1 - \frac{\eta}{100})\varphi_o^* \beta_{lo}^*}{(1 - \frac{\eta}{100})\varphi_o^*}$$

$$\Rightarrow \qquad q\,(1 - \frac{\eta}{100})\,\phi_o^* \leq \sum_{j=1}^{n} \rho_j y_{lj} - (1 - \frac{\eta}{100})\,\phi_o^*\,\beta_{lo}^*$$

$$\Rightarrow \qquad (1 - \frac{\eta}{100})\,\phi_o^* \beta_{lo}^* \leq \sum_{j=1}^{n} \rho_j y_{lj}$$

We define:

$$\bar{\beta}_{ro} = \begin{cases} q + \beta_{lo}^* & r = l \\ \beta_{ro}^* & r \neq l \end{cases}$$

It can be easily seen that $(\lambda^*, \bar{\beta}_o, \rho^*)$ is a feasible solution for the model (10) in which $\bar{\beta}_{lo} > \beta_{lo}^*$. This contradicts point $(\lambda^*, \beta_o^*, \rho^*)$ being Pareto efficient. Therefore, the absurdum is rejected which means that all mentioned constraints are binding. □

Theorem 6. Let $(\bar{\lambda}, \bar{\beta}_o, \bar{\rho})$ is a feasible solution for the model (10), such that:

$$\varphi(\alpha_o, \mu_o, \gamma_o, \bar{\beta}_o) \leq \varphi(X_o, X_o^{"}, X_o', Y_o)$$

Then, $(\bar{\lambda}, \bar{\beta}_o, \bar{\rho})$ is a weak Pareto solution of the model (10).

Proof: Suppose that $(\bar{\lambda}, \bar{\beta}_o, \bar{\rho})$ is not a weak Pareto solution for the model (10), then there is another feasible solution $(\tilde{\lambda}, \tilde{\beta}_o, \tilde{\rho})$ such that:

$$\forall r \qquad \tilde{\beta}_{ro} > \bar{\beta}_{ro}$$

Since $(\tilde{\lambda}, \tilde{\beta}_o, \tilde{\rho})$ is a feasible solution for the model (10), then in terms of the 6th constraint of the model and the absurdum we have:

$$\forall r \qquad \sum_{j=1}^{n} \tilde{\rho}_j y_{rj} \geq (1 - \frac{\eta}{100}) k \varphi_o^* \tilde{\beta}_{ro} > (1 - \frac{\eta}{100}) k \varphi_o^* \bar{\beta}_{ro}$$

So, there exist k>1 such that:

$$\forall r \qquad \sum_{j=1}^{n} \tilde{\rho}_j y_{rj} > (1 - \frac{\eta}{100}) k \varphi_o^* \bar{\beta}_{ro}$$

We define:

$$\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{n+1}) = \begin{cases} \hat{\lambda}_j = \tilde{\lambda}_j & j \neq n+1 \\ \hat{\lambda}_{n+1} = 0 & j = n+1 \end{cases}$$

and

$$\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_{n+1}) = \begin{cases} \hat{\rho}_j = \tilde{\rho}_j & j \neq n+1 \\ \hat{\rho}_{n+1} = 0 & j = n+1 \end{cases}$$

and

$$\hat{\alpha}_o = \alpha_o, \hat{\mu}_o = \mu_o, \hat{\gamma}_o = \gamma_o, k\varphi_o^* = \hat{\varphi}_o, \hat{\beta}_o = \bar{\beta}_o$$

Hence, $(\hat{\lambda}, \hat{\beta}_o, \hat{\rho}, k\varphi_o^*)$ is a feasible solution for the model (11). Then, the objective value for this possible solution is $k\varphi_o^*$. Based on the assumption, we should have $k\varphi_o^* \leq \varphi_o^*$ which results in $k \leq 1$. This contradicts the primary assumption for (k). Therefore, the absurdum is rejected, and the statement is proved. □

Theorem 7. Let DMU_o is efficient and $(\bar{\lambda}, \bar{\beta}_o, \bar{\rho})$ is a weakly efficient solution of the model (10), then:

$$\varphi(\alpha_o, \mu_o, \gamma_o, \bar{\beta}_o) \leq \varphi(X_o, X_o'', X_o', Y_o)$$

Proof: Since DMU_o is efficient, then $\varphi_o^* = 1$. We need to illustrate $\phi_{N+1}^* \leq \phi_o^*$ to achieve the goal, let $\bar{\beta}_o = \tilde{\beta}_o$, $\bar{\lambda} = \tilde{\lambda}$, $\bar{\lambda}_{N+1} = 0$, $\bar{\rho} = \tilde{\rho}$, $\tilde{\rho}_{N+1} = 0$.

Then, we have:

$$\forall r \qquad \sum_{j=1}^{n} \tilde{\rho}_j y_{rj} \geq \varphi_{N+1}^* \bar{\beta}_{ro} > \bar{\beta}_{ro} = \varphi_o^* \bar{\beta}_{ro}$$

Based on the absurdum, suppose that $\phi_{N+1}^* > \phi_o^*$. Since $\varphi_o^* = 1$, we have $\phi_{N+1}^* > 1$.
Hence:

$$\forall r \qquad \sum_{j=1}^{n} \tilde{\rho}_j y_{rj} \geq \varphi_{N+1}^* \bar{\beta}_{ro} > \bar{\beta}_{ro} = \varphi_o^* \bar{\beta}_{ro}$$

For $k > 0$, define:

$$\tilde{\beta}_{ro} = \bar{\beta}_{ro} + k; \ r = 1, \ldots, S$$

So, $(\tilde{\lambda}, \tilde{\beta}_o, \tilde{\rho})$ is a feasible solution for the model (10). But, according to the above definition

$$\tilde{\beta}_{ro} > \bar{\beta}_{ro}; \quad r = 1, \ldots, S$$ □

This contradicts point $(\bar{\lambda}, \bar{\beta}_o, \bar{\rho})$ being a weakly Pareto solution. Hence, the proposition is proved.

4.2. Input Estimation

Two cases are addressed to study the input estimation problem: the first considers the input estimation by improving the efficiency, and the second deals with the input estimation by increasing the shared inputs and maintaining the efficiency. Now, we discuss the first case, i.e., The input estimation with efficiency improvement. In the two-stage DEA structure shown in Figure 1, suppose that outputs of the DMU_o change from Y_o to β_o such that $\beta_o = Y_o + \Delta Y_o$ and $\Delta Y_o \geq 0, \Delta Y_o \neq 0$. In the following, the input vector $(\alpha_o, \mu_o, \gamma_o)$ is estimated so that the efficiency of DMU_o is improved by $\left(1 + \frac{\eta}{100}\right)\%$.

Hence, for the input vector, we have:

$$(\alpha_o, \mu_o, \gamma_o) = (\alpha_{i_1o}, i_1 \in I_1; \ \mu_{i_2o}, i_2 \in I_2; \ \gamma_{ko}, k = 1, \ldots, q) = (X_o, X_o^{"}, X_o') + (\Delta X_o, \Delta X_o^{"}, \Delta X_o')$$

in which $(\Delta X_o, \Delta X_o^{"}, \Delta X_o') \geq 0$. For the next step, following MOLP is considered to approximate the input vector:

$$\begin{aligned}
&\text{Minimize } (\alpha_{i_1o}; \ i_1 \in I_1, \ \mu_{i_2o}; \ i_2 \in I_2, \ \gamma_{ko}; \ k = 1, \ldots, q) \\
&\quad \text{subject to} && && (13) \\
&\quad \sum_{j=1}^{n} \rho_j z_{dj} \leq \left(1 + \tfrac{\eta}{100}\right) \theta_o^* z_{do} && , && d = 1, 2, \ldots, D \\
&\quad \sum_{j=1}^{n} \lambda_j x_{i_1j} \leq \left(1 + \tfrac{\eta}{100}\right) \theta_o^* \alpha_{i_1o} && , && i_1 \in I_1 \\
&\quad \sum_{j=1}^{n} \rho_j x'_{kj} \leq \left(1 + \tfrac{\eta}{100}\right) \theta_o^* \gamma_{ko} && , && k = 1, 2, \ldots, q \\
&\quad \sum_{j=1}^{n} ((1 - \sigma_{i_2j})\rho_j + \sigma_{i_2j}\lambda_j) x^{"}_{i_2j} \leq \left(1 + \tfrac{\eta}{100}\right) \theta_o^* \mu_{i_2o} && , && i_2 \in I_2 \\
&\quad \sum_{j=1}^{n} \lambda_j z_{dj} \geq z_{do} && , && d = 1, 2, \ldots, D \\
&\quad \sum_{j=1}^{n} \rho_j y_{rj} \geq \beta_{ro} && , && r = 1, 2, \ldots, S \\
&\quad \alpha_{i_1o} \geq x_{i_1o} && , && i_1 \in I_1 \\
&\quad \gamma_{ko} \geq x'_{ko} && , && k = 1, 2, \ldots, q \\
&\quad \mu_{i_2o} \geq x^{"}_{i_2o} && , && i_2 \in I_2 \\
&\quad \rho_j, \lambda_j \geq 0 && , && j = 1, 2, \ldots, n
\end{aligned}$$

Note that in the model (12), $0 < \sigma_{i_2j} < 1$, $(j = 1, \ldots, n; \ \ i_2 \in I_2)$ and $0 \leq \eta \leq \frac{100(1-\theta_o^*)}{\theta_o^*}$. It can be seen here that if $\eta = 0$, then the performance level of unit under evaluation will be unchanged with respect to the other units. Besides, if $\eta = \frac{100(1-\theta_o^*)}{\theta_o^*}$, then the unit will be efficient. Note that $(\lambda_j, \rho_j, \alpha_{i_1o}, \mu_{i_2o}, \gamma_{ko})$; $(j = 1, \ldots, n, \ i_1 \in I_1, \ i_2 \in I_2, \ k = 1, \ldots, q)$ is the vector of variables. Furthermore, θ_o^* which was the optimal value of the model (8) improved by $\eta\%$, and β_{ro} are constant values.

Assume that new DMU_o after altering its input and output levels is shown as DMU_{n+1}. Then, the following model taken from the model (8) is applied to evaluate the efficiency of DMU_{n+1}:

$$\begin{aligned}
&\theta_{n+1} = \text{Minimize } \theta \\
&\quad \text{Subject to} && && (14) \\
&\quad \sum_{j=1}^{n} \rho_j z_{dj} + \rho_{n+1} z_{do} \leq \theta z_{do} && , && d = 1, \ldots, D \\
&\quad \sum_{j=1}^{n} \lambda_j x_{i_1j} + \lambda_{n+1} \alpha_{i_1o} \leq \theta \alpha_{i_1o} && , && i_1 \in I_1 \\
&\quad \sum_{j=1}^{n} \rho_j x'_{kj} + \rho_{n+1} \gamma_{ko} \leq \theta \gamma_{ko} && , && k = 1, \ldots, q
\end{aligned}$$

$$\sum_{j=1}^{n}\left(\left(1-\sigma_{i_2j}\right)\rho_j+\sigma_{i_2j}\lambda_j\right)x''_{i_2j}+\left(\left(1-\sigma_{i_2(n+1)}\right)\rho_{n+1}+\sigma_{i_2(n+1)}\lambda_{n+1}\right)\mu_{i_2o}\leq\theta\mu_{i_2o} \qquad i_2\in I_2$$

$$\sum_{j=1}^{n}\lambda_j z_{dj}+\lambda_{n+1}z_{do}\geq z_{do} \quad , \qquad d=1,\dots,D$$

$$\sum_{j=1}^{n}\rho_j y_{rj}+\rho_{n+1}\beta_{ro}\geq\beta_{ro} \quad , \qquad r=1,\dots,S$$

$$\rho_j,\ \lambda_j\geq 0 \quad , \qquad j=1,\dots,n+1$$

Now, consider the models (8) and (14). If in optimality, we have:

$$\theta(\alpha_o,\mu_o,\gamma_o,\beta_o)\geq\theta(X_o,X''_o,X'_o,Y_o)$$

then, the efficiency score is improved. □

in the following the input estimation by increasing the shared inputs and maintaining the efficiency is discussed:

As shown in Figure 1, there are three different types of inputs. Now, in two-stage inverse DEA we aim to answer the following question: If the shared inputs of the first and second stages (x''_{i_2o}) and the inputs of the second stage (x'_{ko}) as well as the outputs (y_{ro}) increase simultaneously, then how much should increase the input of the first stage (x_{i_1o}) such that the efficiency score of DMU_o does not change?

Let outputs of DMU_o change from Y_o to β_o such that $\beta_o=Y_o+\Delta Y_o$ and $\Delta Y_o\geq 0, \Delta Y_o\neq 0$. Also, let the input vector ($X''_o,\ X'_o$) changes to (μ_o,γ_o) such that:

$$(\mu_o,\gamma_o)=(X''_o,\ X'_o)+(\Delta X''_o,\ \Delta X'_o)\ ;\ ((\Delta X''_o,\ \Delta X'_o)\geq\ 0\ \ \&\ (\Delta X''_o,\ \Delta X'_o)\neq\ 0)$$

Therefore, estimating input vector (α_o) is aimed such that efficiency score of DMU_o i.e., θ_o remains unchanged where:

$$\alpha_o=(\alpha_{i_1o};\ i_1\in I_1)=X_o+\Delta X_o\ ;\ \Delta X_o\geq 0$$

If DMU_{n+1} is supposed to be the new unit after changing inputs, and outputs of DMU_o, then model (14) is applied to assess the performance of DMU_{n+1}. It is clear that if the optimal value of the models (8) and (14) is the same, then the value of the efficiency is constant, i.e.:

$$\theta(\alpha_o,\mu_o,\gamma_o,\beta_o)=\theta(X_o,X''_o,X'_o,Y_o)$$

Now, the following MOLP is provided to approximate the input vector:

$$\text{Minimize } (\alpha_{i_1o}\ ;\ i_1\in\ I_1)$$
$$\text{subject to} \tag{15}$$
$$\sum_{j=1}^{n}\rho_j z_{dj}\leq\ \theta_o^* z_{do} \quad , \qquad d=1,2,\dots,D$$

$$
\begin{array}{lll}
\sum_{j=1}^{n} \lambda_j x_{i_1 j} \leq \theta_o^* \alpha_{i_1 o} & , & i_1 \in I_1 \\
\sum_{j=1}^{n} \rho_j x'_{kj} \leq \theta_o^* \gamma_{ko} & , & k = 1, 2, \dots, q \\
\sum_{j=1}^{n} ((1-\sigma_{i_2 j})\rho_j + \sigma_{i_2 j}\lambda_j) x''_{i_2 j} \leq \theta_o^* \mu_{i_2 o} & , & i_2 \in I_2 \\
\sum_{j=1}^{n} \lambda_j z_{dj} \geq z_{do} & , & d = 1, 2, \dots, D \\
\sum_{j=1}^{n} \rho_j y_{rj} \geq \beta_{ro} & , & r = 1, 2, \dots, S \\
\alpha_{i_1 o} \geq x_{i_1 o} & , & i_1 \in I_1 \\
\rho_j , \lambda_j \geq 0 & , & j = 1, 2, \dots, n
\end{array}
$$

Note that θ_o^* as the optimal value of the model (8), and given vectors γ_{ko} , $\mu_{i_2 o}$ & β_{ro} are constant. Furthermore, $(\lambda_j, \rho_j, \alpha_{i_1 o})$ is the vector of variables, and $0 < \sigma_{i_2 j} < 1$; $(j = 1, \dots, n\,;\ \ i_2 \in I_2)$ are as mentioned before.

4.3. Window DEA

As stated in section 3.3, Figure 1 shows the production process used in the model proposed by Jian Feng [5]. Suppose that measuring the performance trend of the DMU under evaluation among a set of DMUs at a specific point in time is aimed such that data are also functions of time. In this case, observing the performance of the DMU under evaluation at the point in time may have a large error in the actual performance. To address this issue, it is recommended that the data be collected and analyzed at different periods when there is a possibility of its change. As illustrated in Figure 2, the aforementioned production process can occur in several periods. This problem is dealt with as window DEA in which it is no longer possible to use the traditional DEA models (Jahanshahloo *et al.,* [37] ,and Emrouznejad and Thanassoulis [38]). Therefore, the inverse two-stage DEA problem should be examined from the window DEA perspective.

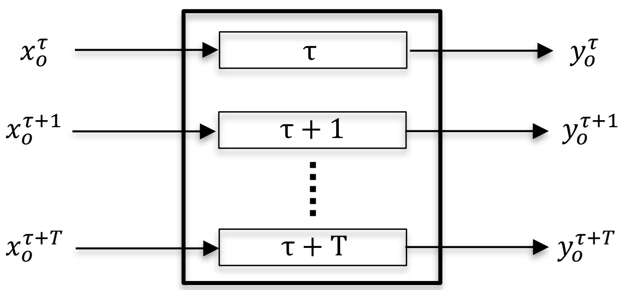

Fig. 2. Multi-period production cycle of DMU_o without internal temporal dependencies

For this purpose, assume that the aim is to assess the performance of the number of DMUs given in an arbitrary time interval (performance window). Since this interval was arbitrarily chosen, without the loss of generality, we can consider the evaluation window as $t = \tau + 1, \dots, \tau + T$. It is imperative to acknowledge that the activity of a decision-maker persists both prior to and subsequent to the arbitrary time interval in which the arbitrary choice is proposed. According to Figure 1, as mentioned earlier, for each DMU in each period, there are three types of inputs (primary inputs, added inputs, and common inputs) and two outputs (middle and final outputs) in which primary and common inputs are shown as $x_{i_1 j}$; $(i_1 \in I_1)$ and $x"_{i_2 j}$; $(i_2 \in I_2)$ respectively and $I_1 \cup I_2 = \{1, \ldots, m\}$ & $I_1 \cap I_2 = \emptyset$.

Furthermore, added inputs are shown as x'_{kj}; $(k = 1, \ldots, q)$. Moreover, assume that each DMU_j; $(j = 1, \ldots, n)$, contains D outputs of the first stage as z_{dj}; $(d = 1, \ldots, D)$ and S ultimate outputs as y_{rj}; $(r = 1, \ldots, S)$ from the second phase.

Furthermore, note that part of z_{dj}; $(d = 1, \ldots, D)$ as the middle outputs are considered input for the second stage, and the rest have entered the market as the final output. The share of intermediate outputs and the share of final outputs are denoted by $\delta_{dj} z_{dj}$ and $(1 - \delta_{dj})\, z_{dj}$, respectively so that $0 < \delta_{dj} < 1$.

Suppose that $x_{i_1 j}^{t=\tau+1,\ldots,\tau+T}$ is a general representation of the initial inputs of each period and $z_{dj}^{t=\tau+1,\ldots,\tau+T}$ express intermediate outputs of each stage where $\tau + 1$ and $\tau + T$ are considered as initial and final periods in τ^{th} evaluation window, respectively. In the production process shown in Figure 2 as the internal structure of Figure 1, it is important to note that the efficiency of a DMU is expressed as the average of the periodic efficiencies in a specified period.

To estimate the relative efficiency in the production procedure in Figure 1, the following input-oriented model is proposed which derived from model (8) and the model presented by Jahanshahloo et al., [37]:

$$\theta_o = \text{Minimize } \frac{\sum_{t=\tau+1}^{t=\tau+T} \theta^t}{T}$$

subject to (16)

$$\sum_{j=1}^{n} \rho_j z_{dj}^t + \delta_d^t = \theta^t z_{do}^t \quad , \quad d = 1, 2, \ldots, D\ ;\ t = \tau+1, \ldots, \tau+T$$

$$\sum_{j=1}^{n} \lambda_j x_{i_1 j}^t + s_{i_1}^t = \theta^t x_{i_1 o}^t \quad , \quad i_1 \in I_1\ ;\ t = \tau+1, \ldots, \tau+T$$

$$\sum_{j=1}^{n} \rho_j (x')_{kj}^t + (s')_k^t = \theta^t (x')_{ko}^t \quad , \quad k = 1, 2, \ldots, q\ ;\ t = \tau+1, \ldots, \tau+T$$

$$\sum_{j=1}^{n} ((1 - \sigma_{i_2 j})\rho_j + \sigma_{i_2 j}\lambda_j)(x")_{i_2 j}^t + (s")_{i_2}^t = \theta^t (x")_{i_2 o}^t \quad , \quad i_2 \in I_2\ ;\ t = \tau+1, \ldots, \tau+T$$

$$\sum_{j=1}^{n} \lambda_j z_{dj}^t + \gamma_d^t = z_{do}^t \quad , \quad d = 1, 2, \ldots, D;\ t = \tau+1, \ldots, \tau+T$$

$$\sum_{j=1}^{n} \rho_j y_{rj}^t + (\gamma')_r^t = y_{ro}^t \quad , \quad r = 1, 2, \ldots, S\ ;\ t = \tau+1, \ldots, \tau+T$$

$$\theta^t \leq 1 \quad , \quad t = \tau+1, \ldots, \tau+T$$

$$\rho_j\ ,\ \lambda_j \geq 0 \quad , \quad j = 1, 2, \ldots, n$$

$$\delta_d^t \geq 0; \quad s_{i_1}^t \geq 0; \quad (s')_k^t \geq 0; \quad (s")_{i_2}^t \geq 0; \quad \gamma_d^t \geq 0 \quad , \quad d = 1, 2, \ldots, D;\ i_1 \in I_1\ ;$$

$(\gamma')_r^t \geq 0$, $k = 1, 2, \ldots, q; i_2 \in I_2$

$r = 1, 2, \ldots, S$;
$t = \tau + 1, \ldots, \tau + T$

In above model, δ_d^t, $s_{i_1}^t$, $(s')_k^t$, $(s")_{i_2}^t$, γ_d^t& $(\gamma')_r^t$ are all auxiliary variables. Now, suppose the evaluation path for DMU_j; $(j = 1, \ldots, n)$ is as follows:

$$(x_j^{\tau+1,\ldots,\tau+T},\ (x")_j^{\tau+1,\ldots,\tau+T},\ (x')_j^{\tau+1,\ldots,\tau+T},\ y_j^{\tau+1,\ldots,\tau+T},\ z_j^{\tau+1,\ldots,\tau+T})$$

Emrouznejad and Thanassoulis (2005) [38] referred to the evaluation path for DMU_o in a multi-period system if any of the following conditions are satisfied:

i. There is currently no evaluation path or combination of assessment paths that, when utilizing the DMU_o-path, yields a higher output in at least one of the evaluation periods without compromising the outputs in certain periods or necessitating an additional input in at least one of the stages.
ii. There are no evaluation pathways or combinations of assessment paths that, when followed via the DMU_o-path, lead to more significant outcomes with less input within a specified evaluation time. This is not possible without increasing the number of entries in other periods or the ability to generate more outputs at any stage.

The following definition is proposed by Emrouznejad and Thanassoulis [38]:

Definition 4. The evaluation path for the DMU_o is efficient if the following relationships satisfy optimality:

$$\theta_o^* = 1\,,\ (\delta_d^t)^* = 0\,,\ \left(s_{i_1}^t\right)^* = 0\,,\ ((s')_k^t)^* = 0\,,\ \left((s")_{i_2}^t\right)^* = 0\,,\ (\gamma_d^t)^* = 0\,,\ ((\gamma')_r^t)^* = 0\,;$$

$$\forall\ (d = 1, 2, \ldots, D\ , i_1\ \in I_1\,, k = 1, 2, \ldots, q, r = 1, 2, \ldots, S,\ i_2\ \in I_2\ \&\ t = \tau + 1, \ldots, \tau + T).$$

5. Numerical Example

Table 1 contains twenty-five DMUs, Including information of Iranian bank branches. First, we describe some definitions regarding the expressions given in the case study:

i. Interests paid: The amount of interest that the bank pays to customers' long-term accounts. Obviously, the higher this number is, the lower the performance of bank.
ii. Personnel score: The score is calculated by combining and adjusting the weighted values of many parameters, including the number of people, executive positions, work experience, compensation, and training hours of staff.
iii. Received Claims: When the bank provides banking facilities to the customers, and the customer is not able to pay his installments at the specified time, in this case, the customer's debt to the bank for the facilities is called the bank's claims. Thus, the amount of collection of these claims by the bank in a period of time is called received claims.
iv. Loan: The amount of money that the bank provides to the customers and the customers repay the mentioned facilities along with the interest based on the type of contract.
v. Profit received: The amount of interest that the bank receives from customers for providing facilities.

As shown in Figure 3, each of these DMUs has three groups of inputs, including inputs of the first stage (Interests paid) X, shared inputs (Personnel score) $X^"$, inputs of the second stage (Received

Claims) X', intermediate products (Loan) Z, outputs (Profit received) Y. Furthermore, the last column denotes the efficiency score obtained from model (2).

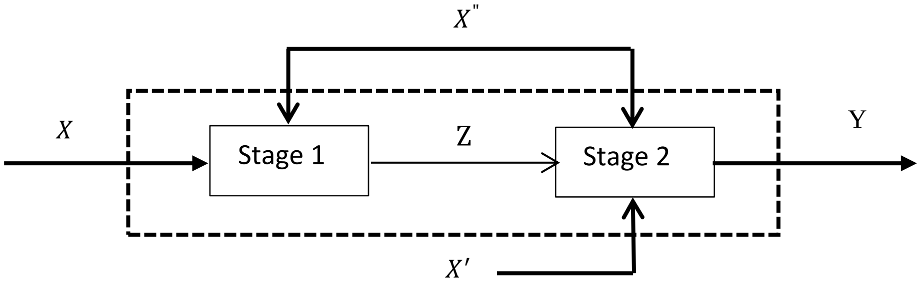

Fig. 3. A Two-stage Network of Iranian bank branches

Following the region's needs, the management of the Bank Branch Office wants to expand the sixth branch and improve its efficiency, so estimating the required output is aimed.

Table 1
Twenty-five DMUs, Including information of Iranian bank branches

DMU_s	X	X'	X''	Z	Y	φ_o^*
DMU_1	2805177638	11994821504	3.84	64212392856	2791562738	1.47996739
DMU_2	3594467903	110059103	5.35	77880596974	4233879876	1.09059259
DMU_3	3544897361	3361995150	9.81	92496206596	6950574909	1.69146388
DMU_4	1637026869	17200888	4.82	10229324293	621536914	1.65522112
DMU_5	1141193835	230962380	6.68	18044217945	1210386464	4.08135735
DMU_6	1298058057	298605509	4.97	6216822256	640204744	6.39624619
DMU_7	1655720390	2517935136	10.97	6723276986	809282030	5.47213389
DMU_8	3269705080	783841829	7.34	50514419171	4442615481	2.28613285
DMU_9	1879114081	96797863	5.57	21579730074	1550192752	2.3926002
DMU_{10}	1057686290	20084929	6.54	14053333853	1314742636	1
DMU_{11}	1440411328	157240192	1.56	29119166915	1585159187	1.1724436
DMU_{12}	2973130019	106858226	4.53	90504304878	4522740900	1
DMU_{13}	3828801690	304483790	7.14	14000129445	9221629290	1
DMU_{14}	1641873024	31582000	6.55	21135376814	1499295285	1.34433106
DMU_{15}	1133793983	1217039129	3.85	51746989236	2233950758	1.15939884
DMU_{16}	877247942	85475000	5.23	29495575267	2236306947	1.48333974
DMU_{17}	4839219824	7716168688	4.87	13798127749	9068971211	1.00216157
DMU_{18}	3188240311	1708846082	12.93	44417322288	3039268874	3.82456206
DMU_{19}	6775382776	985942000	7.2	47467684724	2080344159	3.66549764
DMU_{20}	472101054	708507168	11.17	57863179290	1775227490	1
DMU_{21}	3083576526	2661711908	5.83	60957146147	3746829194	1.84735109
DMU_{22}	3301629838	5721986246	6.52	28131631731	2077376915	4.32897853
DMU_{23}	1830532119	2653205624	3.86	18056815228	1315620133	3.85960441
DMU_{24}	7686691834	8492311825	7.61	68523607568	5835683004	2.2093554
DMU_{25}	1933947316	238079274	9.89	13600845684	933400866	7.87795739

Table 2 compares results obtained from our inverse two-stage models and classic inverse DEA models, which perform as black box for DMU_6. Suppose that inputs of DMU_6, i.e., (129805857, 2986.5509, 4.97) increased to (1408058057, 3586.5509, 7.07) such that the primary efficiency score improved by 30%. In other words, inputs x_6, x_6', and x_6'' were multiplied by 1.085, 1.2, and 1.423, respectively. If the classic inverse DEA model with a black box structure is applied, then the new output is estimated as $\beta_6^* = 12215730959$, which is almost 19.081 times the initial value of Y_6. If our proposed inverse two-stage model (10) is applied for estimating the outputs, then $\beta_6^* = 914578000$ which is almost 1.43 times of Y_6 . It states that our model can be considered efficient to deal with the inverse two-stage problem with common inputs and intermediate products. Because, in many cases, due to the existing limitations on available resources, decision makers cannot increase the outputs 19 times to achieve the desired efficiency. But, by applying our presented model, we only need to increase the Y_6 1.43 times that is a reasonable amount.

Furthermore, by applying model (16), a window DEA problem with t=1,2 was solved for assessing DMU$_6$. Then, we have $\beta_{1,6}^* = 690478000$, $\beta_{2,6}^* = 970521000$ in which $\beta_{1,6}^*$ and $\beta_{2,6}^*$ are the amount of output increase for DMU$_6$ in periods t=1 and t=2, respectively.

Table 2
Comparison between classic, inverse, and window two-stage models for DMU$_6$

Models		$(\alpha_6, \mu_6, \gamma_6)$	β_6^*	Multiples of Y_6
Classic model (3)		(1408058057, 358605509, 7.07)	12215730959	19.08
Proposed model (10)		(1408058057, 358605509, 7.07)	2134015813	1.43
Proposed window model (16)	t=1	(1408058057, 358605509, 7.07)	690478000	Proposed window model (16)
	t=2	(1408058057, 358605509, 7.07)	970521000	

The results illustrate that the static model and the model presented by Jian Feng are not able to give details about what would happen in each stage. It means that, if the process in Figure 1 is repeated for a period of time, the model provided by Jian Feng and static model are not applicable for solving the problem. For example, if we want to assess the performance of a bank for 5 years, the mentioned models are not capable of dealing with the performance of the bank within a model, but by applying the model (16) and considering t=1, 2, …, 5 the decision maker can study the two-stage process within 5 years.

6. Conclusions

An efficient manager should comprehensively understand resource allocation and product supply, drawing upon scientific support. In fact, the decision-maker need to provide appropriate models to efficiently distribute resources and control the product supply for several DMUs in accordance with preset parameters. DEA is widely recognized as a precious approach in this context. However, classic DEA models typically treat the initial inputs to generate the outputs as a singular process, wherein the DMU is seen as a "black box". Consequently, the DMU's internal structure and the production process's middle stages are often disregarded. Therefore, there is no information on which part of a DMU is the cause or source of inefficiency. In contrast, DMUs are represented as systems with two subunits in the two-stage DEA models. Intermediate products are regarded as second stage subunit inputs. Consequently, a two-stage DEA model allows for examining the internal structure of a DMU and its processes. Thus, while emphasizing two issues mentioned above, this paper used the envelopment form of the two-stage models proposed by Jian Feng [5] in the inverse DEA problem for the first time and considered the intermediate productions and input structure simultaneously to provide appropriate tool for analyzing input/output estimation problems. The following issues were also addressed:

- Our presented models discussed estimating all or part of the inputs to improve efficiency, which will help managers, decision-makers, and investors make an expert system.
- The proposed model to estimate the final output to achieve a specified level of efficiency of the stages has the following properties:
  - After estimating the output, the efficiency is not worse than the efficiency with the previous coordinates.
  - The necessary and sufficient conditions for estimating outputs were proved.
- Due to the repetitive processes in some two-stage systems, the presented models were also examined from the viewpoint of window DEA.

The following issues can be mentioned for future research:

- In a network structure, when we have an undesirable final output that is a function of the desirable outputs, how should the output estimation problem be addressed?
- If there is a window network structure that is considered over a period of time, then how the outputs are estimated while the output at the last period is a function of all the inputs from the previous periods?
- The article examined the effect of increasing inputs (excluding the values of intermediate products used as inputs in the subsequent stage) on the output level in the second stage. Subsequent studies might examine the impact of augmenting these inputs on the augmentation of intermediate productions and the consequent rise in final outputs.Since the rate of increase in outputs to achieve a certain level of performance must occur in the future, it is reasonable to treat the number of inputs, and outputs of the unit under evaluation and other units as values of a random variable. Thus, suggesting suitable models for random inputs and outputs appears more valuable than those for specific ones.
- Considering the potential effect of altering primary input on intermediate products, it is recommended to incorporate intermediate products as variables in the output estimation model. The optimal solution can be determined by solving the corresponding nonlinear programming problem.

Author Contributions

Haleh Moradi: Conceptualization, methodology, software, formal analysis, investigation, resources, writing-original draft preparation. Farhad Hosseinzadeh Lotfi: Methodology, validation, writing-review and editing, supervision. Mohsen Rostamy-Malkhalifeh: formal analysis, resources.

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Data Availability Statement

The study did not report any data.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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