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DECISION MAKING UNDER INCOMPLETE DATA: INTUITIONISTIC MULTI FUZZY IDEALS OF NEAR-RING APPROACH

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Abstract. *Real-world data is often partial, uncertain, or incomplete. Decision making based on data as such can be addressed by fuzzy sets and related systems. This article studies the intuitionistic multi-fuzzy sub-near rings and Intuitionistic multi-fuzzy ideals of near rings. It presents some of the elementary operations and relations defined on these structures. The concept of level subsets and support of the Intuitionistic multi-fuzzy sub-near ring is also presented. It looks into and demonstrated a few characteristics of intuitionistic multi-fuzzy near-rings and ideals. This research advances fuzzy set theory, which is often applied to problems involving pattern recognition and multiple criterion decision-making. Thus, the results may be beneficial to artificial intelligence related research. Alternatively, the intuitionistic multi-fuzzy approach may be applied to vector spaces and modules or extended to inter-valued fuzzy systems.*

Keywords: *Intuitionistic Fuzzy Set, Near-ring, Fuzzy Multi Near-ring, Intuitionistic multi fuzzy Near-ring, Ideals, fuzzy multi ring*

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1. Introduction

The concept of "fuzzy sets" was initially proposed by Zadeh (Zadeh, 1965), which opened the door for set theory researchers. Many versions and generalizations of fuzzy sets have appeared to solve problems such as multi-criteria decision-making, recognition of patterns and diagnosis of diseases (Broumi, Ajay, et al., 2022; Ashraf et al., 2022; Broumi, Sundareswaran, et al., 2022) waste management (Zhumadillayeva et al., 2020) and voltage balancing (Taghieh et al., 2022). Applications of the fuzzy systems to some other areas can be studied (Gulzar, Alghazzawi, et al., 2020; Gulzar, Mateen, et al., 2020; Kausar, 2019; Kausar et al., 2020; Riaz et al., 2022b). Due to the appearance of fuzzy sets and their associated systems as problem-solving tools in MCDM (Abbas et al., 2021; Abdullah, 2013; Kahraman, 2008) and other disciplines of our daily life, more people were attracted towards this area of research. Theoretical mathematicians use this idea to generalize well-known mathematical structures. For example, Rosenfeld used it to develop a fuzzy group structure and generalize the classical group. This was the initial development of fuzzy group theory. Liu introduced fuzzy rings and studied various properties of rings and ideals in a fuzzy context. Within a few years, the fuzzification of algebraic structures became a hot topic within the research community. Researchers developed more fuzzy algebraic structures like fuzzy modules, fuzzy algebras, fuzzy sub near rings etc. were developed [see (Al-Husban, 2021; Fathi & Salleh, 2009; Hur et al., 2005; Rahman & Saikia, 2012; Zhan & Ma, 2005)]. Salah Abuzaid was the first who introduced the notion of fuzzy sub near-rings (Abou-Zaid, 1991) and later, its variants were studied(Asif et al., 2020; Hussain et al., 2022).

The concept of multisets is initiated by Yager (Yager, 1986). The multi-fuzzy groups were proposed by T. K. Shinoj et al. [see (Dresher & Ore, 1938; Shinoj et al., 2015)]. They studied the basic properties of multi-fuzzy groups and presented a few preliminary results. The fuzzy versions of multi-subrings and their ideals were established by L. Sujatha (Sujatha, 2014) in 2014. She also proved that the finite multi-fuzzy subrings (ideals) intersection is a multi-fuzzy subring (ideal). Fuzzy multi-near-rings and their associated multi-ideals were introduced by Tahan et al. (Al Tahan et al., 2021) in 2021. They defined various operations on multi-ideals of fuzzy near-rings. They presented foundational results related to fuzzy multi sub-near-rings, fuzzy multi-ideals and the operations defined on multi-ideals. The anti-fuzzy multi-ideals of near-rings were considered by Hoskova (Hoskova-Mayerova & Al Tahan, 2021).

One of the well-known generalizations of fuzzy sets is the intuitionistic fuzzy set proposed by Atanassov(Atanassov, 1986). Renowned mathematicians also use this set to fuzzify algebraic structures. Fathi was the first to describe the notion intuitionistic fuzzy group (Fathi & Salleh, 2009) Consequently, the intuitionistic versions of groups, rings, ideals, modules, near-rings etc., have been established (Hur et al., 2005). Many researchers (Kausar & Waqar, 2019; Kousar et al., 2021, 2022; Riaz et al., 2022a) used intuitionistic fuzzy sets and developed different structures. These included triangular intuitionistic fuzzy linear programming, lattice-valued intuitionistic fuzzy subgroup type-3, algebraic codes over lattice-valued intuitionistic fuzzy type-3 submodules, codes over lattice-valued intuitionistic fuzzy set type-3, non-associative ordered semi-groups by intuitionistic fuzzy bi ideals. Others further developed different versions of intuitionistic fuzzy sets, such as complex intuitionistic fuzzy sets in group theory and t-intuitionistic fuzzy subgroups(Gulzar, Alghazzawi, et al., 2020; Gulzar, Mateen, et al., 2020). It also encompasses the finite intuitionistic anti-fuzzy normal subrings' direct product (Kausar, 2019) intuitionistic fuzzy normal subrings (Shah et al., 2012).

The primary purpose of this study is to find the connections between multi-fuzzy near-rings and intuitionistic fuzzy sets. For this, we first define intuitionistic fuzzy multi-near-rings (IFMNRs) and ideals associated with this structure. We also define the basic operations and study the critical properties of this structure. Moreover, we study the support and $(\propto, \beta)-$ level subsets of IFMNRs and produce a few related results. This work contributes to the fuzzy set theory and fuzzy algebra, which are extensively used to solve multi-criteria decision-making and pattern recognition problems (Kaharman et. al., 2008).

2. Motivation and Scope

Fuzzy sets and associated systems deal with theoretical and practical problems equipped with incomplete, uncertain, or ambiguous data. Although this is a very young discipline started with the work of L. A. Zadeh but gained the attention of many researchers quickly. The reason is that it can be directly applied to theoretical and daily life problems, including Multi-criteria decision-making, Pattern recognition, disease diagnosis and Management. So, it is worth studying the fuzzy systems to produce more sufficient and adequate theoretical bases used to develop better problem-solving tools. This article also proposes a fuzzy algebraic system intuitionistic fuzzy multi-near-rings, a generalization of well-known fuzzy multi-near-rings.

3. Preliminaries

This study first recalls some basic definitions of fuzzy and multi-fuzzy sets.

Definition 2.1: Let $\mathcal{N}$ be a non-empty set, then a fuzzy set $\tilde{\mathcal{A}}$ is given by an ordered pair (Zadeh, 1965)

$$\tilde{\mathcal{A}} = \{(\mathrm{s}, \Omega_{\tilde{\mathcal{A}}}(s)) \,/\, s \,\epsilon \mathcal{N}\}$$

$\Omega_{\tilde{\mathcal{A}}}$ is the degree of membership, and $\Omega_{\tilde{\mathcal{A}}} : \mathcal{N} \rightarrow [0,1]$ is the membership function.

Example 1. If $\mathcal{N} = \{s, t, \mathrm{z}\}$ then $\tilde{\mathcal{A}} = \{(s, 0.51), (\mathrm{t}, 0.03), (\mathrm{z}, 0.21)\}$ be fuzzy set of $\mathcal{N}$.

Definition 2.2: Let $\mathcal{N}$ be a set that is not empty. An intuitionistic fuzzy set will be (Atanassov, 1986)

$$\tilde{\mathcal{A}} = \{< s,\ \Omega_{\tilde{\mathcal{A}}}(s), \mho_{\tilde{\mathcal{A}}}(s) > /\ s \,\epsilon \mathcal{N}\}$$

Where $\Omega_{\tilde{\mathcal{A}}}$ and $\mho_{\tilde{\mathcal{A}}}$ are the degrees of membership and non-membership function, respectively, they are defined as $\Omega_{\tilde{\mathcal{A}}} : \mathcal{N} \rightarrow [0,1]$ and $\mho_{\tilde{\mathcal{A}}} : \mathcal{N} \rightarrow [0,1]$.

For each $\epsilon\, \mathcal{N}$ $0 \leq \Omega_{\tilde{\mathcal{A}}}(s) + \mho_{\tilde{\mathcal{A}}}(s) \leq 1$.

Remark 1. Every fuzzy set is an intuitionistic fuzzy set.

Definition 2.3: Let $\mathcal{N}$ be a non-empty set, then $\mathcal{M}$ be a multiset drawn from $\mathcal{N}$, characterized by a count function $\mathcal{CM}: \mathcal{N} \rightarrow \mathbb{N}$, where $\mathcal{CM}$ represent the number of repetition of an element in $\mathcal{M}$ and $\mathbb{N}$ is set of positive integers (Yager, 1986; Hoskova-Mayerova & Al Tahan, 2021).

Let $\mathcal{N} = \{s_1, s_2, s_3, \ldots, s_n\}$ be a set; then a multiset $\mathcal{M}$ will be represented as $\mathcal{M} = \{\mathfrak{m}_1/s_1, \mathfrak{m}_2/s_2, \mathfrak{m}_3/s_3, \ldots, \mathfrak{m}_n/s_n\}$ where $\mathfrak{m}_i = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots, \mathfrak{m}_n\}$ represents the number of repetition of an element in $\mathcal{M}$.

Example 2. Let $\mathcal{N} = \{s, t, z\}$ be a set, then $\mathcal{M} = \{s, t, t, s, z, t, t\}$ is a multiset, and it can also be represented as $\mathcal{M} = \{2/s, 4/t, 1/z,\}$.

Definition 2.4: Per Al Tahan et al. (2021), let $\mathcal{N}$ be a non-empty set then a fuzzy multiset constructed from $\mathcal{N}$ can be represented as;

$$\mathcal{A} = \{(s, \mathcal{CM}_{\mathcal{A}}(s)) \;/\; s \in \mathcal{N}$$

Where $\mathcal{CM}_{\mathcal{A}}: \mathcal{N} \to Q$ represents the count membership function, here Q is a set of all crisp multisets which is constructed from unit interval [0,1] and for each $s \in \mathcal{N}$, $\mathcal{CM}_{\mathcal{A}}(s)$ is a decreasingly ordered sequence that is $\Omega^1_{\mathcal{A}}(s) \geq \Omega^2_{\mathcal{A}}(s) \geq \Omega^3_{\mathcal{A}}(s) \geq \cdots \geq \Omega^n_{\mathcal{A}}(s)$.

Definition 2.5: Let $\mathcal{N}$ be a non-empty set. An intuitionistic version of a fuzzy multiset can be represented;

$$\mathcal{A} = \{< s, \mathcal{CM}_{\mathcal{A}}(s), \mathcal{CN}_{\mathcal{A}}(s) > / \; s \in \mathcal{N}\}$$

$\mathcal{CMA}: \mathcal{N} \to Q$ represents the count membership function, and $\mathcal{CN}_{\mathcal{A}}: \mathcal{N} \to Q$ shows the count non-membership function. Here Q is a set of all crisp multisets which is constructed from unit interval [0,1], and for each $s \in \mathcal{N}$, $\mathcal{CM}_{\mathcal{A}}(s)$ is a decreasingly ordered sequence that is $\Omega^1_{\mathcal{A}}(s) \geq \Omega^2_{\mathcal{A}}(s) \geq \Omega^3_{\mathcal{A}}(s) \geq \cdots \geq \Omega^n_{\mathcal{A}}(s)$ and $\mathcal{CN}_{\mathcal{A}}(s)$ is denoted by $(\mho^1_{\mathcal{A}}(s), \mho^2_{\mathcal{A}}(s), \mho^3_{\mathcal{A}}(s), \ldots, \mho^n_{\mathcal{A}}(s))$. For each $s \in \mathcal{N}, 0 \leq \mathcal{CM}_{\mathcal{A}}(s) + \mathcal{CN}_{\mathcal{A}}(s) \leq 1$.

Remark 2. $\mathcal{CM}_{\mathcal{A}}(s)$ is ordered decreasingly, but the corresponding $\mathcal{CN}_{\mathcal{A}}(s)$ may need to be in decreasing and increasing order.

Remark 3. An intuitionistic fuzzy set on set $\mathcal{N}$ can be treated as a particular case of an intuitionistic fuzzy multiset, if $\mathcal{CM}_{\mathcal{A}}(s) = \Omega^1_{\mathcal{A}}(s). \mathcal{CN}_{\mathcal{A}}(s) = \mho^1_{\mathcal{A}}(s) \; \forall \; s \in \mathcal{N}$.

Example 3. Let $\mathcal{N} = \{a, b, c\}$. Then, $\mathcal{A}$ is an intuitionistic fuzzy multiset over $\mathcal{N}$ with count functions:

$$\mathcal{CM}_{\mathcal{A}}(s) = \begin{cases} 0.4, 0.4, 0.6 & if\ s = a \\ 0.1, 0.1, 0.1 & if\ s = b \\ 1, 1, 0.5 & if\ s = c \end{cases}$$

$$\mathcal{CN}_{\mathcal{A}}(s) = \begin{cases} 0.6, 0.6, 0.4 & if\ s = a \\ 0.9, 0.9, 0.9 & if\ s = b \\ 0.5 & if\ s = c \end{cases}$$

Definition 2.6: According to Al Tahan et. al (2021), Shinoj et. al., (2015), let $\mathcal{N} \neq \emptyset$ be a set, $\mathcal{A}$ and $\mathcal{B}$ be two intuitionistic fuzzy multisets over $\mathcal{N}$ with fuzzy count functions $\mathcal{CM}_{\mathcal{A}}(s)$, $\mathcal{CM}_{\mathcal{B}}(s)$ and $\mathcal{CN}_{\mathcal{A}}(s)$, $\mathcal{CN}_{\mathcal{B}}(s)$ respectively, then:

$\mathcal{A} \subseteq \mathcal{B}$ if $\mathcal{CM}_{\mathcal{A}}(s) \leq \mathcal{CM}_{\mathcal{B}}(s)$ and $\mathcal{CN}_{\mathcal{A}}(s) \leq \mathcal{CN}_{\mathcal{A}}(s) \; \forall \; s \in \mathcal{N}$

$\mathcal{A} = \mathcal{B}$ if $\mathcal{CM}_{\mathcal{A}}(s) = \mathcal{CM}_{\mathcal{B}}(s)$ and $\mathcal{CN}_{\mathcal{A}}(s) = \mathcal{CN}_{\mathcal{A}}(s) \; \forall \; s \in \mathcal{N}$

$\mathcal{A} \cap \mathcal{B}$ is defined as;

$\mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s) = \min\{\mathcal{CM}_{\mathcal{A}}(s), \mathcal{CM}_{\mathcal{B}}(s)\}$ and $\mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s) = \max\{\mathcal{CN}_{\mathcal{A}}(s), \mathcal{CN}_{\mathcal{B}}(s)\}$

$\mathcal{A} \cup \mathcal{B}$ is defined as;

$\mathcal{CM}_{\mathcal{A}\cup\mathcal{B}}(s) = \max\{\mathcal{CM}_{\mathcal{A}}(s), \mathcal{CM}_{\mathcal{B}}(s)\}$ and $\mathcal{CN}_{\mathcal{A}\cup\mathcal{B}}(s) = \min\{\mathcal{CN}_{\mathcal{A}}(s), \mathcal{CN}_{\mathcal{B}}(s)\}$

The complement of an intuitionistic fuzzy multiset is defined as;

$$\mathcal{A}^c = \{< s, \mathcal{CN}_{\mathcal{A}}(s), \mathcal{CM}_{\mathcal{A}}(s) > / \; s \in \mathcal{N}\}$$

Example 4. Let $\mathcal{N} = \{a, b, c, d\}$ and $\mathcal{A}, \mathcal{B}$ be two intuitionistic fuzzy multisets over $\mathcal{N}$ that is;

$\mathcal{A} = \{< a, (0.8,0.8,0.6), (0.2,0.2,0.4) >, < b, (0.7,0.7,0.7), (0.3,0.3,0.3) >, < c, (1,0.9), (0.1) >\}$,
$\mathcal{B} = \{< a, (1,0.5,0.5), (0.5,0.5) >, < c, (0.6,0.6), (0.4,0.4) >\}$ Then;
$\mathcal{A} \cap \mathcal{B} = \{< a, (0.8,0.5,0.5), (0.2,0.5,0.5) >, < c, (0.6,0.6), (0.4,0.4) >\}$
And;
$\mathcal{A} \cup \mathcal{B} = \{< a, (1,0.8,0.6), (0.2,0.4) >, < c, (1,0.9), (0.1) >\}$.

Definition 2.7: Let $\mathcal{N}$ be a non-empty set, then $(\mathcal{N}, +, \cdot)$ is said to be left (right) NR if (Al Tahan et. al., 2021, Asif, et. al., 2020, Hussain et. al., 2022);
$(\mathcal{N}, +)$ is a group;
$(\mathcal{N}, \cdot)$ is a semi-group;
$\mathcal{N}$ Satisfies left (right) distributive law, that is;
$s \cdot (t + z) = s \cdot t + s \cdot z \; \forall \; s, t, z \in \mathcal{N}$ (Left distributive)
$(t + z) \cdot s = t \cdot s + z \cdot s \; \forall \; s, t, z \in \mathcal{N}$ (Right distributive)

Example 5. Let $\mathbb{R}$ be a set of real numbers, then $(\mathbb{R}, +, \cdot)$ form NR under standard " + " and multiplication is defined as $a \cdot b = a \; \forall \; a, b \in \mathbb{R}$.

Note: Throughout this text, we write NR instead of near-ring and NRs for near-rings.

Definition 2.8: Let $(\mathcal{N}, +, \cdot)$ be a NR, and $\mathcal{J}$ be a sub- NR of $\mathcal{N}$ then $\mathcal{J}$ is said to be an ideal of $\mathcal{N}$ (Al Tahan et. al., 2021, Asif, et. al., 2020, Hussain et. al., 2022), if;

i. $s + t - s \in \mathcal{J} \; \forall \; s \in \mathcal{N} \; t \in \mathcal{J}$
ii. $s \cdot t \in \mathcal{J} \; \forall \; s \in \mathcal{N} \wedge t \in \mathcal{J}$
iii. $(s + a) \cdot t - s \cdot t \in \mathcal{J} \; \forall \; s, t \in \mathcal{N} \wedge a \in \mathcal{J}$

Remark 4 If $\mathcal{J}$ fulfils conditions (i) and (ii), then $\mathcal{J}$ is said to be the left ideal, and if $\mathcal{J}$ satisfies (i) and (iii), then $\mathcal{J}$ is said to be the ideal of $\mathcal{N}$. If $\mathcal{J}$ is the left and right ideal of $\mathcal{N}$, then $\mathcal{J}$ is said to be an ideal of $\mathcal{N}$.

Example 6. Let $\mathcal{M}_2(\mathbb{Z})$ be the set of all possible 2 by 2 matrices with entries from $\mathbb{Z}$ and $\mathcal{M}_2(\mathbb{Z})$ form NR under standard addition and multiplication is defined as $(a_{ij}) \cdot (b_{ij}) = (a_{ij})$ for some $(a_{ij}), (b_{ij}) \in \mathcal{M}_2(\mathbb{Z})$ then $\mathcal{M}_2(2\mathbb{Z})$ is right ideal of $\mathcal{M}_2(\mathbb{Z})$.

Definition 2.9: (Abou-Zaid, 1991) Let $(\mathcal{N}, +, \cdot)$ be a NR and $\mathcal{A}$ be fuzzy set over $\mathcal{N}$ then $\mathcal{A}$ is said to be fuzzy sub-NR of $\mathcal{N}$ if $\forall \; s, t \in \mathcal{N}$ the following condition satisfies;

I. $\Omega_{\mathcal{A}}(s - t) \geq \Omega_{\mathcal{A}}(s) \wedge \Omega_{\mathcal{A}}(t)$
II. $\Omega_{\mathcal{A}}(s \cdot t) \geq \Omega_{\mathcal{A}}(s) \wedge \Omega_{\mathcal{A}}(t)$

Definition 2.10: (Abou-Zaid, 1991) Let $(\mathcal{N}, +, \cdot)$ be a NR and $\mathcal{A}$ be fuzzy set over $\mathcal{N}$ then $\mathcal{A}$ is said to be the fuzzy ideal of NR if $\forall \; s, t \in \mathcal{N}$ the following condition satisfies;

I. $\Omega_{\mathcal{A}}(s - t) \geq \Omega_{\mathcal{A}}(s) \wedge \Omega_{\mathcal{A}}(t)$
II. $\Omega_{\mathcal{A}}(s \cdot t) \geq \Omega_{\mathcal{A}}(s) \wedge \Omega_{\mathcal{A}}(t)$
III. $\Omega_{\mathcal{A}}(s + t - s) \geq \Omega_{\mathcal{A}}(t)$
IV. $\Omega_{\mathcal{A}}(s \cdot t) \geq \Omega_{\mathcal{A}}(t)$
V. $\Omega_{\mathcal{A}}((s + a)t - st) \geq \Omega_{\mathcal{A}}(a) \; \forall \; a \in \mathcal{N}$

Definition 2.11: (Al Tahan et. al., 2021) Let $(\mathcal{N}, +, \cdot)$ be a NR and $\mathcal{A}$ be fuzzy multiset over $\mathcal{N}$ then $\mathcal{A}$ is said to be fuzzy multi sub-NR of $\mathcal{N}$ if $\forall \; s, t \in \mathcal{N}$ the following condition satisfies;

I. $\mathcal{CM}_{\mathcal{A}}(s - t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)$
II. $\mathcal{CM}_{\mathcal{A}}(s \cdot t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)$

Definition 2.12: (Al Tahan et. al., 2021) Let $(\mathcal{N},+,\cdot)$ be a NR and $\mathcal{A}$ be a fuzzy multiset over $\mathcal{N}$ then $\mathcal{A}$ is said to be a fuzzy multi ideal of NR $\mathcal{N}$ if $\forall\, s,t \in \mathcal{N}$ the following condition satisfies;

III. $\mathcal{CM}_{\mathcal{A}}(s-t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)$

IV. $\mathcal{CM}_{\mathcal{A}}(s \cdot t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)$

V. $\mathcal{CM}_{\mathcal{A}}(s+t-s) \geq \mathcal{CM}_{\mathcal{A}}(t)$

VI. $\mathcal{CM}_{\mathcal{A}}(s \cdot t) \geq \mathcal{CM}_{\mathcal{A}}(t)$

VII. $\mathcal{CM}_{\mathcal{A}}((s+a)t-st) \geq \mathcal{CM}_{\mathcal{A}}(a)\ \forall\, a \in \mathcal{N}$

4. Main Result

Definition 3.1: Let $(\mathcal{N},+,\cdot)$ be a NR. An intuitionistic multi-fuzzy set $\mathcal{A}$ is an intuitionistic multi-fuzzy sub-NR over $\mathcal{N}$. If $\forall\, s,t \in \mathcal{N}$, the following conditions are satisfied;

I. $\mathcal{CM}_{\mathcal{A}}(s-t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)$ and $\mathcal{CN}_{\mathcal{A}}(s-t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t)$

II. $\mathcal{CM}_{\mathcal{A}}(s \cdot t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)$ and $\mathcal{CN}_{\mathcal{A}}(s \cdot t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t)$

Where $\mathcal{CM}_{\mathcal{A}}$ and $\mathcal{CN}_{\mathcal{A}}$ count membership and count non-membership functions, respectively.

Example 7. Let $(\mathbb{Z},+,\cdot)$ be a NR under standard addition and multiplication. Then, A is an intuitionistic multi-fuzzy sub-NR over $\mathbb{Z}$ with fuzzy count functions given by;

$$\mathcal{CM}_{\mathcal{A}} = \begin{cases} (0.21,0.21,0.21) & \text{if } s \text{ even number} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{CN}_{\mathcal{A}} = \begin{cases} (0.79,0.79,0.79) & \text{if } s \text{ even number} \\ 1 & \text{otherwise} \end{cases}$$

Definition 3.2: Let $(\mathcal{N},+,\cdot)$ be a NR. An intuitionistic multi-fuzzy set $\mathcal{A}$ is considered an intuitionistic fuzzy multi-ideal of $\mathcal{N}$. If $\forall\, s,t \in \mathcal{N}$, the following conditions are satisfied;

$\mathcal{CM}_{\mathcal{A}}(s-t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)$ and $\mathcal{CN}_{\mathcal{A}}(s-t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t)$

$\mathcal{CM}_{\mathcal{A}}(s \cdot t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)$ and $\mathcal{CN}_{\mathcal{A}}(s \cdot t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t)$

$\mathcal{CM}_{\mathcal{A}}(s+t-s) \geq \mathcal{CM}_{\mathcal{A}}(t)$ and $\mathcal{CN}_{\mathcal{A}}(s+t-s) \leq \mathcal{CN}_{\mathcal{A}}(t)$

$\mathcal{CM}_{\mathcal{A}}(s \cdot t) \geq \mathcal{CM}_{\mathcal{A}}(t)$ and $\mathcal{CN}_{\mathcal{A}}(s \cdot t) \leq \mathcal{CN}_{\mathcal{A}}(t)$

$\mathcal{CM}_{\mathcal{A}}((s+a)t-st) \geq \mathcal{CM}_{\mathcal{A}}(a)$ and

$\mathcal{CN}_{\mathcal{A}}\big((s+a)t-st\big) \leq \mathcal{CN}_{\mathcal{A}}(a) \quad \forall\, a \in \mathcal{N}$

Example 8. Let $(\mathbb{Z},+,\cdot)$ be a NR under standard addition and multiplication. Then, A is an intuitionistic multi-fuzzy ideal of $\mathbb{Z}$ with fuzzy count functions given by;

$$\mathcal{CM}_{\mathcal{A}} = \begin{cases} (1,0.9,0.7,0.5,0.4) & \text{if } s \text{ is a multiple of } 3 \\ (0.9,0.6,0.3) & \text{otherwise} \end{cases}$$

$$\mathcal{CN}_{\mathcal{A}} = \begin{cases} (0.1,0.3,0.5,0.6) & \text{if } s \text{ is multiple of } 3 \\ (0.1,0.4,0.7) & \text{otherwise} \end{cases}$$

Remark 5. Every intuitionistic multi-fuzzy ideal of a NR. $\mathcal{N}$ is an intuitionistic multi-fuzzy sub-NR of $\mathcal{N}$.

This study then demonstrates some exciting results for intuitionistic multi-fuzzy near-rings, which are proven for other algebraic structures (Al-Tahan et. al., 2021).

Proposition 3.3: Let $(\mathcal{N},+,\cdot)$ be a NR and $\mathcal{A},\mathcal{B}$ be two intuitionistic multi-fuzzy sub-NRs of $\mathcal{N}$, then $\mathcal{A} \cap \mathcal{B}$ is also an intuitionistic multi-fuzzy sub NR of $\mathcal{N}$.

Proof. Let $\mathcal{A},\mathcal{B}$ be two intuitionistic multi-fuzzy sub-NRs of $\mathcal{N}$, then $\forall\ s,t \in \mathcal{N}$, we are to show conditions of definition 3.1.

$\mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s-t) = \mathcal{CM}_{\mathcal{A}}(s-t) \wedge \mathcal{CM}_{\mathcal{B}}(s-t)$
$\geq \{\mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)\} \wedge \{\mathcal{CM}_{\mathcal{B}}(s) \wedge \mathcal{CM}_{\mathcal{B}}(t)\}$
$= \{\mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{B}}(s)\} \wedge \{\mathcal{CM}_{\mathcal{A}}(t) \wedge \mathcal{CM}_{\mathcal{B}}(t)\}$
$= \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s) \wedge \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(t) \Rightarrow \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s-t)$
$\geq \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s) \wedge \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(t)$

And, $\mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s-t) = \mathcal{CN}_{\mathcal{A}}(s-t) \vee \mathcal{CN}_{\mathcal{B}}(s-t) \leq \{\mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t)\} \vee \{\mathcal{CN}_{\mathcal{B}}(s) \vee \mathcal{CN}_{\mathcal{B}}(t)\} = \{\mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{B}}(s)\} \vee \{\mathcal{CN}_{\mathcal{A}}(t) \vee \mathcal{CN}_{\mathcal{B}}(t)\} = \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s) \vee \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(t) \Rightarrow \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s-t) \leq \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s) \vee \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(t)$

Also, $\mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s\cdot t) = \mathcal{CM}_{\mathcal{A}}(s\cdot t) \wedge \mathcal{CM}_{\mathcal{B}}(s\cdot t) \geq \{\mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)\} \wedge \{\mathcal{CM}_{\mathcal{B}}(s) \wedge \mathcal{CM}_{\mathcal{B}}(t)\} = \{\mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{B}}(s)\} \wedge \{\mathcal{CM}_{\mathcal{A}}(t) \wedge \mathcal{CM}_{\mathcal{B}}(t)\} = \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s) \wedge \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(t) \Rightarrow \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s-t) \geq \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s) \wedge \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(t)$

And, $\mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s\cdot t) = \mathcal{CN}_{\mathcal{A}}(s\cdot t) \vee \mathcal{CN}_{\mathcal{B}}(s\cdot t) \leq \{\mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t)\} \vee \{\mathcal{CN}_{\mathcal{B}}(s) \vee \mathcal{CN}_{\mathcal{B}}(t)\} = \{\mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{B}}(s)\} \vee \{\mathcal{CN}_{\mathcal{A}}(t) \vee \mathcal{CN}_{\mathcal{B}}(t)\} = \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s) \vee \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(t) \Rightarrow \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s\cdot t) \leq \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s) \vee \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(t)$

Corollary 1. Let $(\mathcal{N},+,\cdot)$ be a NR, and $\mathcal{A}_i$ be an intuitionistic multi-fuzzy sub-NRs of $\mathcal{N}$ for $i = 1,2,3,\dots n$, then $\cap_{i=1}^{n}\ \mathcal{A}_i$ is also an intuitionistic multi-fuzzy sub-NR of $\mathcal{N}$.

Proposition 3.4: (Al-Tahan et. al., 2021 & Shinoj et. al., 2015) Let $(\mathcal{N},+,\cdot)$ be a NR, and $\mathcal{A}\ and\ \mathcal{B}$ be intuitionistic multi-fuzzy ideals of $\mathcal{N}$. Then, $\mathcal{A}\cap\mathcal{B}$ is also an intuitionistic multi-fuzzy ideal of $\mathcal{N}$.

Proof. Let $\mathcal{A},\mathcal{B}$ be two intuitionistic multi-fuzzy ideals of $\mathcal{N}$, then $\forall\ s,t \in \mathcal{N}$, we are to show conditions of definition 3.2. The first two have been done in proposition 3.3.

$\Rightarrow \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s+t-s) = \mathcal{CM}_{\mathcal{A}}(s+t-s) \wedge \mathcal{CM}_{\mathcal{B}}(s+t-s) \geq \mathcal{CM}_{\mathcal{A}}(t) \wedge \mathcal{CM}_{\mathcal{B}}(t)$

Since $\mathcal{A}$ and $\mathcal{B}$ are IMFI's $\Rightarrow \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s-t) \geq \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(t)$

And, $\mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s+t-s) = \mathcal{CN}_{\mathcal{A}}(s+t-s) \vee \mathcal{CN}_{\mathcal{B}}(s+t-s) \leq \mathcal{CN}_{\mathcal{A}}(t) \vee \mathcal{CN}_{\mathcal{B}}(t)$

$\Rightarrow \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s-t) \leq \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(t)$

Moreover, $\mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s\cdot t) = \mathcal{CM}_{\mathcal{A}}(s\cdot t) \wedge \mathcal{CM}_{\mathcal{B}}(s\cdot t) \geq \mathcal{CM}_{\mathcal{A}}(t) \wedge \mathcal{CM}_{\mathcal{B}}(t)$

$\Rightarrow \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(s\cdot t) \geq \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(t)$

Also, $\mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s\cdot t) = \mathcal{CN}_{\mathcal{A}}(s\cdot t) \vee \mathcal{CN}_{\mathcal{B}}(s\cdot t) \leq \mathcal{CN}_{\mathcal{A}}(t) \vee \mathcal{CN}_{\mathcal{B}}(t)$

$\Rightarrow \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(s\cdot t) \leq \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(t)$

Let $a \in \mathcal{U}$ then $\mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}((s+a)t-st) = \mathcal{CM}_{\mathcal{A}}((s+a)t-st) \wedge \mathcal{CM}_{\mathcal{B}}((s+a)t-st) \geq \mathcal{CM}_{\mathcal{A}}(a) \wedge \mathcal{CM}_{\mathcal{B}}(a)$

Because $\mathcal{A}$ and $\mathcal{B}$ are IMFI's $\Rightarrow \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}((s+a)t-st) \geq \mathcal{CM}_{\mathcal{A}\cap\mathcal{B}}(a)$

And, $\mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}((s+a)t-st) = \mathcal{CN}_{\mathcal{A}}((s+a)t-st) \vee \mathcal{CN}_{\mathcal{B}}((s+a)t-st) \leq \mathcal{CN}_{\mathcal{A}}(a) \vee \mathcal{CN}_{\mathcal{B}}(a) \Rightarrow \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}((s+a)t-st) \leq \mathcal{CN}_{\mathcal{A}\cap\mathcal{B}}(a)$ ■

Corollary 2. Let $(\mathcal{N},+,\cdot)$ be a NR, and $\mathcal{A}_i$ be the intuitionistic multi-fuzzy ideal of NRs of $\mathcal{N}$ for $i = 1,2,3,\dots,n$ then $\cap_{i=1}^{n}\ \mathcal{A}_i$ is also an intuitionistic multi-fuzzy ideal of $\mathcal{N}$.

Example 9. Let $(Z_3,+,\cdot)$ be a NR and $\mathcal{A},\mathcal{B}$ be two intuitionistic multi-fuzzy ideals of Z_3 given by;

$\mathcal{A} = \{< 0, (0.9,0.9,0.5,0.5), (0.1,0.1,0.5,0.5) >, < 1, (0.7,0.3,0.1,0.1), (0.3,0.7,0.9,0.9) >, < 2, (0.7,0.3,0.1,0.1), (0.3,0.7,0.9,0.9) >\}$

And

$\mathcal{B} = \{< 0, (0.7,0.7,0.7), (0.3,0.3,0.3) >, < 1, (0.5,0.5,0.2), (0.5,0.5,0.8) >,$
$< 2, (0.5,0.5,0.2), (0.5,0.5,0.8) >\}$

$\mathcal{A} \cap \mathcal{B} = \{< 0, (0.7,0.7,0.5), (0.3,0.3,0.5) >, < 1, (0.5,0.3,0.1), (0.5,0.7,0.9) >,$
$< 2, (0.5,0.3,0.1), (0.5,0.7,0.9) >\}$

It also satisfies the conditions of definition 3.2 and forms an intuitionistic multi-fuzzy ideal of Z_3.

Remark 6. Let $(\mathcal{N}, +,\cdot)$ be a NR and $\mathcal{A}, \mathcal{B}$ be two intuitionistic multi-fuzzy ideals of $\mathcal{N}$. Then, $\mathcal{A} \cup \mathcal{B}$ may or may not be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$.

Example 10. Let $(\mathbb{Z}, +,\cdot)$ be a NR and $\mathcal{A}, \mathcal{B}$ be two intuitionistic multi-fuzzy ideals of $\mathbb{Z}$ given by;

$$\mathcal{CM}_{\mathcal{A}}(s) = \begin{cases} (0.6,0.6) & \text{if } s \text{ is multiple of 5} \\ (0.3,0.2) & \text{otherwise} \end{cases}$$

$$\mathcal{CN}_{\mathcal{A}}(s) = \begin{cases} (0.4,0.4) & \text{if } s \text{ is multiple of 5} \\ (0.7,0.8) & \text{otherwise} \end{cases}$$

And,

$$\mathcal{CM}_{\mathcal{B}}(s) = \begin{cases} (0.5,0.4) & \text{if } 2/s \\ (0.4,0.3) & \text{otherwise} \end{cases}$$

$$\mathcal{CN}_{\mathcal{B}}(s) = \begin{cases} (0.5,0.6) & \text{if } 2/s \\ (0.6,0.7) & \text{otherwise} \end{cases}$$

Then,

$$\mathcal{CM}_{\mathcal{A}\cup\mathcal{B}}(s) = \begin{cases} (0.6,0.6) & \text{if } s \text{ is multiple of 5 and } 2/s \\ (0.5,0.4) & \text{if } 2/s \\ (0.4,0.3) & \text{otherwise} \end{cases}$$

$$\mathcal{CN}_{\mathcal{A}\cup\mathcal{B}}(s) = \begin{cases} (0.4,0.4) & \text{if } s \text{ is multiple of 5 and } 2/s \\ (0.5,0.6) & \text{if } 2/s \\ (0.6,0.7) & \text{otherwise} \end{cases}$$

So, $\mathcal{CM}_{\mathcal{A}\cup\mathcal{B}}(6-5) \ngeq \mathcal{CM}_{\mathcal{A}\cup\mathcal{B}}(6) \wedge \mathcal{CM}_{\mathcal{A}\cup\mathcal{B}}(5)$ M Vble number ot of definition 3.2 and form intuitionistic fuzzy multi ideal of $\mathcal{CN}_{\mathcal{A}\cup\mathcal{B}}(6-5) \nleq \mathcal{CM}_{\mathcal{A}\cup\mathcal{B}}(6) \vee \mathcal{CM}_{\mathcal{A}\cup\mathcal{B}}(5)$.

It does not satisfy the conditions of definition 3.2 and does not form the intuitionistic multi-fuzzy ideal of $\mathbb{Z}$.

Proposition 3.5: Let $(\mathcal{N}, +,\cdot)$ be a NR, and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$ (Shinoj et. al., 2015), then;

$\mathcal{CM}_{\mathcal{A}}(s) \leq \mathcal{CM}_{\mathcal{A}}(0)$ and $\mathcal{CN}_{\mathcal{A}}(s) \geq \mathcal{CN}_{\mathcal{A}}(0)$ $\forall\, s \in \mathcal{N}$

$\mathcal{CM}_{\mathcal{A}}(-s) = \mathcal{CM}_{\mathcal{A}}(s)$ and $\mathcal{CN}_{\mathcal{A}}(-s) = \mathcal{CN}_{\mathcal{A}}(s)$ $\forall\, s \in \mathcal{N}$

$\mathcal{CM}_{\mathcal{A}}(s+t-s) = \mathcal{CM}_{\mathcal{A}}(t)$ and $\mathcal{CN}_{\mathcal{A}}(s+t-s) = \mathcal{CN}_{\mathcal{A}}(t)$ $\forall\, s,t \in \mathcal{N}$

Proof 1). Let $s \in \mathcal{N}$, then we have

$\mathcal{CM}_{\mathcal{A}}(0) = \mathcal{CM}_{\mathcal{A}}(s-s) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(s) \geq \mathcal{CM}_{\mathcal{A}}(s)$

Also, $\mathcal{CN}_{\mathcal{A}}(0) = \mathcal{CN}_{\mathcal{A}}(s-s) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(s) \leq \mathcal{CN}_{\mathcal{A}}(s)$

Proof 2). Let $s \in \mathcal{N}$, then we have

$\mathcal{CM}_{\mathcal{A}}(-s) = \mathcal{CM}_{\mathcal{A}}(0-s) \geq \mathcal{CM}_{\mathcal{A}}(0) \wedge \mathcal{CM}_{\mathcal{A}}(s) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(s) = \mathcal{CM}_{\mathcal{A}}(s)$

Also, $\mathcal{CN}_{\mathcal{A}}(-s) = \mathcal{CN}_{\mathcal{A}}(0-s) \leq \mathcal{CN}_{\mathcal{A}}(0) \vee \mathcal{CN}_{\mathcal{A}}(s) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(s) = \mathcal{CN}_{\mathcal{A}}(s)$

Proof 3). Is straight forward

Proposition 3.6: Let $(\mathcal{N},+,\cdot)$ be a NR, and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$ if;

$\mathcal{CM}_{\mathcal{A}}(s-t)=\mathcal{CM}_{\mathcal{A}}(0) \Rightarrow \mathcal{CM}_{\mathcal{A}}(s)=\mathcal{CM}_{\mathcal{A}}(t)$ and

$\mathcal{CN}_{\mathcal{A}}(s-t)=\mathcal{CN}_{\mathcal{A}}(0) \Rightarrow \mathcal{CN}_{\mathcal{A}}(s)=\mathcal{CN}_{\mathcal{A}}(t)$

Proof. Let $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$ and $\mathcal{CM}_{\mathcal{A}}(s-t)=\mathcal{CM}_{\mathcal{A}}(0)$ $\forall\, s,t \in \mathcal{N}$ then,

$\mathcal{CM}_{\mathcal{A}}(s)=\mathcal{CM}_{\mathcal{A}}(s-t+t) \geq \mathcal{CM}_{\mathcal{A}}(s-t) \wedge \mathcal{CM}_{\mathcal{A}}(t) \geq \mathcal{CM}_{\mathcal{A}}(0) \wedge \mathcal{CM}_{\mathcal{A}}(t)$

Using proposition 3.5

$,\mathcal{CM}_{\mathcal{A}}(s)=\mathcal{CM}_{\mathcal{A}}(t.)$

Also, suppose that $\mathcal{CN}_{\mathcal{A}}(s-t)=\mathcal{CM}_{\mathcal{A}}(0)\ \forall\, s,t \in \mathcal{N}$. Then,

$\mathcal{CN}_{\mathcal{A}}(s)=\mathcal{CN}_{\mathcal{A}}(s-t+t) \leq \mathcal{CN}_{\mathcal{A}}(s-t) \vee \mathcal{CN}_{\mathcal{A}}(t) \leq \mathcal{CN}_{\mathcal{A}}(0) \vee \mathcal{CN}_{\mathcal{A}}(t)$

$\Rightarrow \mathcal{CN}_{\mathcal{A}}(s)=\mathcal{CN}_{\mathcal{A}}(t)$

Definition 3.7: Let $(\mathcal{N},+,\cdot)$ be a NR, and $\mathcal{A}$ be an intuitionistic multi-fuzzy set of $\mathcal{N}$, then $\mathcal{A}^*$ is defined as $\mathcal{A}^*=\{\, s \in \mathcal{N}$ such that $\mathcal{CM}_{\mathcal{A}}(s)=\mathcal{CM}_{\mathcal{A}}(0)$ and $\mathcal{CN}_{\mathcal{A}}(s)=\mathcal{CN}_{\mathcal{A}}(0)\}$.

Example 11. Let $(\mathbb{Z},+,\cdot)$ be a NR and $\mathcal{A},\mathcal{B}$ be two intuitionistic multi-fuzzy ideals of $\mathbb{Z}$ defined in example 10

$.\mathcal{Then}, \mathcal{A}^* = 5\mathbb{Z}$

Example 12. Let $(Z_3,+,\cdot)$ be a NR and $\mathcal{B}$ be an intuitionistic multi-fuzzy ideal of Z_3 defined in example 9.$\mathcal{Then}, \mathcal{B}^*=\emptyset$

Lemma3.8: Let $(\mathcal{N},+,\cdot)$ be a NR, and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$ then;

$$\mathcal{CM}_{\mathcal{A}}(s) \leq \mathcal{CM}_{\mathcal{A}}(0) \text{ and } \mathcal{CN}_{\mathcal{A}}(s) \geq \mathcal{CN}_{\mathcal{A}}(0)\ \forall\, s \in \mathcal{N}$$

Proof. The proof is done in proposition 3.5.

Proposition 3.9: Let $(\mathcal{N},+,\cdot)$ be a NR and $\mathcal{A}$ be an intuitionistic multi-fuzzy sub-NR of $\mathcal{N}$, then $\mathcal{A}^*$ is also sub-NR of $\mathcal{N}$ (Shinoj et. al., 2015) .

Proof. Let $s,t \in \mathcal{A}^*$then,

$\mathcal{CM}_{\mathcal{A}}(s)=\mathcal{CM}_{\mathcal{A}}(0)$ and $\mathcal{CN}_{\mathcal{A}}(s)=\mathcal{CN}_{\mathcal{A}}(0)$

$\mathcal{CM}_{\mathcal{A}}(t)=\mathcal{CM}_{\mathcal{A}}(0)$ and $\mathcal{CN}_{\mathcal{A}}(t)=\mathcal{CN}_{\mathcal{A}}(0)$

$\mathcal{CM}_{\mathcal{A}}(s-t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)=\mathcal{CM}_{\mathcal{A}}(0)$ and that $\mathcal{CM}_{\mathcal{A}}(s) \leq \mathcal{CM}_{\mathcal{A}}(0)\ \forall\, s \in \mathcal{N}$

$\Rightarrow \mathcal{CM}_{\mathcal{A}}(s-t)=\mathcal{CM}_{\mathcal{A}}(0)$

Also, $\mathcal{CN}_{\mathcal{A}}(s-t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t)=\mathcal{CN}_{\mathcal{A}}(0)$ and that $\mathcal{CN}_{\mathcal{A}}(s) \geq \mathcal{CN}_{\mathcal{A}}(0)\ \forall\, s \in \mathcal{N}$

$\Rightarrow \mathcal{CN}_{\mathcal{A}}(s-t)=\mathcal{CN}_{\mathcal{A}}(0)$

hence $s-t \in \mathcal{A}^*$

Moreover, $\mathcal{CM}_{\mathcal{A}}(s\cdot t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t)=\mathcal{CM}_{\mathcal{A}}(0)$ and that $\mathcal{CM}_{\mathcal{A}}(s) \leq \mathcal{CM}_{\mathcal{A}}(0)\ \forall\, s \in \mathcal{N}$

$\Rightarrow \mathcal{CM}_{\mathcal{A}}(s\cdot t)=\mathcal{CM}_{\mathcal{A}}(0)$

Also, $\mathcal{CN}_{\mathcal{A}}(s\cdot t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t)=\mathcal{CN}_{\mathcal{A}}(0)$ and that $\mathcal{CN}_{\mathcal{A}}(s) \geq \mathcal{CN}_{\mathcal{A}}(0)\ \forall\, s \in \mathcal{N}$

$\Rightarrow \mathcal{CN}_{\mathcal{A}}(s\cdot t)=\mathcal{CN}_{\mathcal{A}}(0)$

hence $s\cdot t \in \mathcal{A}^*$

Proposition 3.10: (Al-Tahan et. al. 2021) Let $(\mathcal{N},+,\cdot)$ be a NR, and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$, then $\mathcal{A}^*$ is also an ideal of $\mathcal{N}$.

Proof. Let $s \epsilon \mathcal{A}^*$ and $a, b \epsilon \mathcal{N}$ then,
$\mathcal{CM}_{\mathcal{A}}(s) = \mathcal{CM}_{\mathcal{A}}(0)$ and $\mathcal{CN}_{\mathcal{A}}(s) = \mathcal{CN}_{\mathcal{A}}(0)$
$\Rightarrow \mathcal{CM}_{\mathcal{A}}(a + s - a) \geq \mathcal{CM}_{\mathcal{A}}(s) = \mathcal{CM}_{\mathcal{A}}(0)$ and that $\mathcal{CM}_{\mathcal{A}}(s) \leq \mathcal{CM}_{\mathcal{A}}(0)\ \forall\ s \epsilon \mathcal{N}$
$\Rightarrow \mathcal{CM}_{\mathcal{A}}(a + s - a) = \mathcal{CM}_{\mathcal{A}}(0)$
Also, $\mathcal{CN}_{\mathcal{A}}(a + s - a) \leq \mathcal{CN}_{\mathcal{A}}(s) = \mathcal{CN}_{\mathcal{A}}(0)$ and that $\mathcal{CN}_{\mathcal{A}}(s) \geq \mathcal{CN}_{\mathcal{A}}(0)\ \forall\ s \epsilon \mathcal{N}$
$\Rightarrow \mathcal{CN}_{\mathcal{A}}(a + s - a) = \mathcal{CN}_{\mathcal{A}}(0)$
hence$(a + s - a) \epsilon \mathcal{A}^*$
Moreover, $\mathcal{CM}_{\mathcal{A}}(a \cdot s) \geq \mathcal{CM}_{\mathcal{A}}(s) = \mathcal{CM}_{\mathcal{A}}(0)$ and that $\mathcal{CM}_{\mathcal{A}}(s) \leq \mathcal{CM}_{\mathcal{A}}(0)\ \forall\ s \epsilon \mathcal{N}$

$\Rightarrow \mathcal{CM}_{\mathcal{A}}(a \cdot s) = \mathcal{CM}_{\mathcal{A}}(0)$
Also, $\mathcal{CN}_{\mathcal{A}}(a \cdot s) \leq \mathcal{CN}_{\mathcal{A}}(s) = \mathcal{CN}_{\mathcal{A}}(0)$ and that $\mathcal{CN}_{\mathcal{A}}(s) \geq \mathcal{CN}_{\mathcal{A}}(0)\ \forall\ s \epsilon \mathcal{N}$
$\Rightarrow \mathcal{CN}_{\mathcal{A}}(a \cdot s) = \mathcal{CN}_{\mathcal{A}}(0)$
hence $a \cdot s \epsilon \mathcal{A}^*$
Now, $\mathcal{CM}_{\mathcal{A}}\big((a + s)b - ab\big) \geq \mathcal{CM}_{\mathcal{A}}(s) = \mathcal{CM}_{\mathcal{A}}(0)$ and that $\mathcal{CM}_{\mathcal{A}}(s) \leq \mathcal{CM}_{\mathcal{A}}(0)\ \forall\ s \epsilon \mathcal{N}$
$\Rightarrow \mathcal{CM}_{\mathcal{A}}((a + s)b - ab) = \mathcal{CM}_{\mathcal{A}}(0)$
Also $\mathcal{CN}_{\mathcal{A}}\big((a + s)b - ab\big) \leq \mathcal{CN}_{\mathcal{A}}(s) = \mathcal{CN}_{\mathcal{A}}(0)$ and that $\mathcal{CN}_{\mathcal{A}}(s) \geq \mathcal{CN}_{\mathcal{A}}(0)\ \forall\ s \epsilon \mathcal{N}$
$\Rightarrow \mathcal{CN}_{\mathcal{A}}((a + s)b - ab) = \mathcal{CN}_{\mathcal{A}}(0)$
Hence $((a + s)b - ab) \epsilon \mathcal{A}^*$

Definition 3.11: Let $\mathcal{N}$ be a non-empty set, and $\mathcal{A}$ be an intuitionistic multi-fuzzy set of $\mathcal{N}$; then the support of $\mathcal{N}$ can be defined as $\mathcal{A}_* = \{s \epsilon \mathcal{N}$ such that $\mathcal{CM}_{\mathcal{A}}(s) > 0$ and $\mathcal{CN}_{\mathcal{A}}(s) < 1\}$.

Example 13. Let $(\mathbb{Z}, +, \cdot)$ be a NR under standard addition and multiplication, and $\mathcal{A}$ be an intuitionistic multi-fuzzy sub-NR over $\mathbb{Z}$ defined in example 7, then,
$\mathcal{A}_* = 2\,\mathbb{Z}$

Proposition 3.12: Let $(\mathcal{N}, +, \cdot)$ be a NR, and $\mathcal{A}$ be an intuitionistic multi-fuzzy sub NR of $\mathcal{N}$, then $\mathcal{A}_*$ is also sub NR of $\mathcal{N}$.

Proof. Let $s, t \epsilon \mathcal{A}_*$then, $\mathcal{CM}_{\mathcal{A}}(s) > 0$ and $\mathcal{CN}_{\mathcal{A}}(s) < 1$ also $\mathcal{CM}_{\mathcal{A}}(t) > 0$ and $\mathcal{CN}_{\mathcal{A}}(t) < 1$
$\Rightarrow \mathcal{CM}_{\mathcal{A}}(s - t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t) > 0 \Rightarrow \mathcal{CM}_{\mathcal{A}}(s - t) > 0$
Also, $\mathcal{CN}_{\mathcal{A}}(s - t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t) < 1 \Rightarrow \mathcal{CN}_{\mathcal{A}}(s - t) < 1$
hence $s - t \epsilon \mathcal{A}_*$
Moreover, $\mathcal{CM}_{\mathcal{A}}(s \cdot t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t) > 0 \Rightarrow \mathcal{CM}_{\mathcal{A}}(s \cdot t) > 0$

Also, $\mathcal{CN}_{\mathcal{A}}(s \cdot t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t) < 1$
$\Rightarrow \mathcal{CN}_{\mathcal{A}}(s \cdot t) < 1$ hence $s \cdot t \epsilon \mathcal{A}_*$

Proposition 3.13: Let $(\mathcal{N}, +, \cdot)$ be a NR, and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$, then $\mathcal{A}_*$ is also an ideal of $\mathcal{N}$.

Proof. Let $s \epsilon \mathcal{A}_*$ and $a, b \epsilon \mathcal{N}$ then $\mathcal{CM}_{\mathcal{A}}(s) > 0$ and $\mathcal{CN}_{\mathcal{A}}(s) < 1$
$\Rightarrow \mathcal{CM}_{\mathcal{A}}(a + s - a) \geq \mathcal{CM}_{\mathcal{A}}(s) > 0 \Rightarrow \mathcal{CM}_{\mathcal{A}}(a + s - a) > 0$
Also, $\mathcal{CN}_{\mathcal{A}}(a + s - a) \leq \mathcal{CN}_{\mathcal{A}}(s) < 1 \Rightarrow \mathcal{CN}_{\mathcal{A}}(a + s - a) < 1$
hence $(a + s - a) \epsilon \mathcal{A}_*$
Moreover, $\mathcal{CM}_{\mathcal{A}}(a \cdot s) \geq \mathcal{CM}_{\mathcal{A}}(s) > 0 \Rightarrow \mathcal{CM}_{\mathcal{A}}(a \cdot s) > 0$

Also, $\mathcal{CN}_{\mathcal{A}}(a \cdot s) \leq \mathcal{CN}_{\mathcal{A}}(s) < 1 \Rightarrow \mathcal{CN}_{\mathcal{A}}(a \cdot s) < 1$

hence $a \cdot s \in \mathcal{A}_*$
Now, $\mathcal{CM}_{\mathcal{A}}\big((a+s)b-ab\big) \geq \mathcal{CM}_{\mathcal{A}}(s) > 0 \Rightarrow \mathcal{CM}_{\mathcal{A}}((a+s)b-ab) > 0$
Also $\mathcal{CN}_{\mathcal{A}}\big((a+s)b-ab\big) \leq \mathcal{CN}_{\mathcal{A}}(s) < 1 \Rightarrow \mathcal{CN}_{\mathcal{A}}((a+s)b-ab) < 1$
Hence $((a+s)b-ab) \in \mathcal{A}_*$

Definition 3.14: Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty sets, A and $\mathcal{B}$ be two intuitionistic fuzzy multisets of $\mathcal{U}$ and $\mathcal{V}$, respectively, then $\mathcal{A} \times \mathcal{B}$ can be defined as,
$\mathcal{A} \times \mathcal{B}(u, v) = \{< (u, v), \mathcal{CM}_{\mathcal{A}}(u) \wedge \mathcal{CM}_{\mathcal{B}}(v), \mathcal{CN}_{\mathcal{A}}(u) \vee \mathcal{CN}_{\mathcal{B}}(v) >\}$
Proposition 3.15: Let $\mathcal{U}$ and $\mathcal{V}$ be two NRs and $\mathcal{A}, \mathcal{B}$ be intuitionistic multi-fuzzy sub-NR of $\mathcal{U}, \mathcal{V}$ respectively, then $\mathcal{A} \times \mathcal{B}$ is also an intuitionistic multi-fuzzy sub-NR of $\mathcal{U} \times \mathcal{V}$.
Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two intuitionistic multi-fuzzy sub-NRs and $(u_1, v_1), (u_2, v_2) \in \mathcal{U} \times \mathcal{V}$. Then we are to show the conditions of Definition 3.1.
(I)
$$\begin{aligned}\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1-u_2),(v_1-v_2)) &= \mathcal{CM}_{\mathcal{A}}(u_1-u_2) \wedge \mathcal{CM}_{\mathcal{B}}(v_1-v_2)\\ &\geq \{\mathcal{CM}_{\mathcal{A}}(u_1) \wedge \mathcal{CM}_{\mathcal{A}}(u_2)\} \wedge \{\mathcal{CM}_{\mathcal{B}}(v_1) \wedge \mathcal{CM}_{\mathcal{B}}(v_2)\}\\ &= \{\mathcal{CM}_{\mathcal{A}}(u_1) \wedge \mathcal{CM}_{\mathcal{B}}(v_1)\} \wedge \{\mathcal{CM}_{\mathcal{A}}(u_2) \wedge \mathcal{CM}_{\mathcal{B}}(v_2)\}\\ &= \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_1, v_1) \wedge \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)\end{aligned}$$
$\Rightarrow \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1-u_2),(u_1-v_2)) \geq \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_1, v_1) \wedge \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$
And,
$$\begin{aligned}\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1-u_2),(v_1-v_2)) &= \mathcal{CN}_{\mathcal{A}}(u_1-u_2) \vee \mathcal{CN}_{\mathcal{B}}(v_1-v_2)\\ &\leq \{\mathcal{CN}_{\mathcal{A}}(u_1) \vee \mathcal{CN}_{\mathcal{A}}(u_2)\} \vee \{\mathcal{CN}_{\mathcal{B}}(v_1) \vee \mathcal{CN}_{\mathcal{B}}(v_2)\}\\ &= \{\mathcal{CN}_{\mathcal{A}}(u_1) \vee \mathcal{CN}_{\mathcal{B}}(v_1)\} \vee \{\mathcal{CN}_{\mathcal{A}}(u_2) \vee \mathcal{CN}_{\mathcal{B}}(v_2)\}\\ &= \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_1, v_1) \vee \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)\end{aligned}$$
$\Rightarrow \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1-u_2),(v_1-v_2)) \leq \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_1, v_1) \vee \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$
(II)
$$\begin{aligned}\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1u_2),(v_1v_2)) &= \mathcal{CM}_{\mathcal{A}}(u_1u_2) \wedge \mathcal{CM}_{\mathcal{B}}(v_1v_2)\\ &\geq \{\mathcal{CM}_{\mathcal{A}}(u_1) \wedge \mathcal{CM}_{\mathcal{A}}(u_2)\} \wedge \{\mathcal{CM}_{\mathcal{B}}(v_1) \wedge \mathcal{CM}_{\mathcal{B}}(v_2)\}\\ &= \{\mathcal{CM}_{\mathcal{A}}(u_1) \wedge \mathcal{CM}_{\mathcal{B}}(v_1)\} \wedge \{\mathcal{CM}_{\mathcal{A}}(u_2) \wedge \mathcal{CM}_{\mathcal{B}}(v_2)\}\\ &= \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_1, v_1) \wedge \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)\end{aligned}$$
$\Rightarrow \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1u_2),(v_1v_2)) \geq \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_1, v_1) \wedge \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$
And,
$$\begin{aligned}\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1u_2),(v_1v_2)) &= \mathcal{CN}_{\mathcal{A}}(u_1u_2) \vee \mathcal{CN}_{\mathcal{B}}(v_1v_2)\\ &\leq \{\mathcal{CN}_{\mathcal{A}}(u_1) \vee \mathcal{CN}_{\mathcal{A}}(u_2)\} \vee \{\mathcal{CN}_{\mathcal{B}}(v_1) \vee \mathcal{CN}_{\mathcal{B}}(v_2)\}\\ &= \{\mathcal{CN}_{\mathcal{A}}(u_1) \vee \mathcal{CN}_{\mathcal{B}}(v_1)\} \vee \{\mathcal{CN}_{\mathcal{A}}(u_2) \vee \mathcal{CN}_{\mathcal{B}}(v_2)\}\\ &= \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_1, v_1) \vee \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)\end{aligned}$$
$\Rightarrow \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1u_2),(v_1v_2)) \leq \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_1, v_1) \vee \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$
Corollary 3. Let $\mathcal{N}_i$ be NRs and $\mathcal{A}_i$ be intuitionistic multi-fuzzy sub-NR of $\mathcal{N}_i$ for $i = 1,2,\dots,n$, then $\prod_{i=1}^{n} \mathcal{A}_i$ is an intuitionistic multi-fuzzy sub-NR of $\prod_{i=1}^{n} \mathcal{N}_i$ where $\mathcal{CM}_{\mathcal{A}_1\times\mathcal{A}_2\times\dots\times\mathcal{A}_n}(n_1, n_2, \dots, n_n) = \bigwedge_{i=1}^{n} \mathcal{CM}_{\mathcal{A}_i}(n_i)$ and $\mathcal{CN}_{\mathcal{A}_1\times\mathcal{A}_2\times\dots\times\mathcal{A}_n}(n_1, n_2, \dots, n_n) = \bigvee_{i=1}^{n} \mathcal{CN}_{\mathcal{A}_i}(n_i)\ \forall\, n_i \in \mathcal{N}_i$.
Proposition 3.16: Let $\mathcal{U}$ and $\mathcal{V}$ be two NRs, and $\mathcal{A}\ and\ \mathcal{B}$ be the intuitionistic multi-fuzzy ideal of $\mathcal{U}$ and $\mathcal{V}$, respectively. Then, $\mathcal{A} \times \mathcal{B}$ is also an intuitionistic multi-fuzzy ideal of $\mathcal{U} \times \mathcal{V}$ (Al-Tahan et. al., 2021 & Sujatha, 2014).

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two intuitionistic multi-fuzzy sub-NRs and $(u_1, v_1), (u_2, v_2) \epsilon\ \mathcal{U} \times \mathcal{V}$, then we are to show conditions of Definition 3.2. (I) and (II) has been done in proposition 3.15 then;

$\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1 + u_2 - u_1), (v_1 + v_2 - v_1)) = \mathcal{CM}_{\mathcal{A}}(u_1 + u_2 - u_1) \wedge \mathcal{CM}_{\mathcal{B}}(v_1 + v_2 - v_1) \geq \mathcal{CM}_{\mathcal{A}}(u_2) \wedge \mathcal{CM}_{\mathcal{B}}(v_2) = \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)\ \therefore\ \mathcal{A}, \mathcal{B}$ are $IMFI's$

$\Rightarrow \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1 + u_2 - u_1), (v_1 + v_2 - v_1)) \geq \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$

and $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1 + u_2 - u_1), (v_1 + v_2 - v_1))$

$= \mathcal{CN}_{\mathcal{A}}(u_1 + u_2 - u_1) \vee \mathcal{CN}_{\mathcal{B}}(v_1 + v_2 - v_1)$

$\leq \mathcal{CN}_{\mathcal{A}}(u_2) \vee \mathcal{CN}_{\mathcal{B}}(v_2) = \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$

$\Rightarrow \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1 + u_2 - u_1), (v_1 + v_2 - v_1)) \leq \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$

Also, $\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1u_2), (v_1v_2)) = \mathcal{CM}_{\mathcal{A}}(u_1u_2) \wedge \mathcal{CM}_{\mathcal{B}}(v_1v_2) \geq \mathcal{CM}_{\mathcal{A}}(u_2) \wedge \mathcal{CM}_{\mathcal{B}}(v_2) = \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)\ \therefore\ \mathcal{A}, \mathcal{B}$ are $IMFI's$

$\Rightarrow \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1u_2), (v_1v_2)) \geq \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$

And, $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1u_2), (v_1v_2)) = \mathcal{CN}_{\mathcal{A}}(u_1u_2) \vee \mathcal{CN}_{\mathcal{B}}(v_1v_2) \leq \mathcal{CN}_{\mathcal{A}}(u_2) \vee \mathcal{CN}_{\mathcal{B}}(v_2) = \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$

$\Rightarrow \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1u_2), (v_1v_2)) \leq \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(u_2, v_2)$

Let $n_1, n_2 \epsilon\ \mathcal{N}$ then,

$\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1 + n_1)u_2 - u_1u_2), (v_1 + n_2)v_2 - v_1v_2))$

$= \mathcal{CM}_{\mathcal{A}}((u_1 + n_1)u_2 - u_1u_2) \wedge \mathcal{CM}_{\mathcal{B}}((v_1 + n_2)v_2 - v_1v_2)$

$\geq \mathcal{CM}_{\mathcal{A}}(n_1) \wedge \mathcal{CM}_{\mathcal{B}}(n_2) = \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(n_1, n_2)$

$\Rightarrow \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}((u_1 + n_1)u_2 - u_1u_2), (v_1 + n_2)v_2 - v_1v_2)) \geq \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(n_1, n_2)$

And $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1 + n_1)u_2 - u_1u_2), (v_1 + n_2)v_2 - v_1v_2)) = \mathcal{CN}_{\mathcal{A}}((u_1 + n_1)u_2 - u_1u_2) \vee \mathcal{CN}_{\mathcal{B}}((v_1 + n_2)v_2 - v_1v_2) \leq \mathcal{CN}_{\mathcal{A}}(n_1) \vee \mathcal{CN}_{\mathcal{B}}(n_2) = \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(n_1, n_2)$

$\Rightarrow \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}((u_1 + n_1)u_2 - u_1u_2), (v_1 + n_2)v_2 - v_1v_2)) \leq \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(n_1, n_2)$

Corollary 4. Let $\mathcal{N}_i$ be NRs and $\mathcal{A}_i$ be intuitionistic multi-fuzzy ideals of $\mathcal{N}_i$ for $i = 1,2, \ldots, n$, then $\prod_{i=1}^{n} \mathcal{A}_i$ is an intuitionistic multi-fuzzy ideal of $\prod_{i=1}^{n} \mathcal{N}_i$ where $\mathcal{CM}_{\mathcal{A}_1\times\mathcal{A}_2\times\ldots\times\mathcal{A}_n}(n_1, n_2, \ldots, n_n) = \wedge_{i=1}^{n} \mathcal{CM}_{\mathcal{A}_i}(n_i)$ and $\mathcal{CN}_{\mathcal{A}_1\times\mathcal{A}_2\times\ldots\times\mathcal{A}_n}(n_1, n_2, \ldots, n_n) = \vee_{i=1}^{n} \mathcal{CN}_{\mathcal{A}_i}(n_i)\ \forall\ n_i\ \epsilon\ \mathcal{N}_i$.

Example 14. Let $(\mathbb{Z}, +, \cdot)$ be a NR and $\mathcal{A}, \mathcal{B}$ be two intuitionistic multi-fuzzy ideals of $\mathbb{Z}$ defined in example 10. Then $\mathcal{A} \times \mathcal{B}$ is an intuitionistic multi-fuzzy ideal of $\mathbb{Z} \times \mathbb{Z}$ with count functions given by;

$$\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s,t) = \begin{cases} (0.5,0.4) & \text{if } s \text{ is multiple of 5 and } 2/t \\ (0.4,0.3) & \text{if } s \text{ is multiple of 5} \\ (0.3,0.2) & \text{otherwise} \end{cases}$$

$$\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s,t) = \begin{cases} (0.5,0.6) & \text{if } s \text{ is multiple of 5 and } 2/t \\ (0.6,0.7) & \text{if } s \text{ is multiple of 5} \\ (0.7,0.8) & \text{otherwise} \end{cases}$$

Definition 3.17: Let $\mathcal{N}$ be a non-empty set and $\mathcal{A}$ be an intuitionistic multi-fuzzy set of $\mathcal{N}$ and $\theta = \{\theta_1, \theta_2, \theta_3, \ldots, \theta_n\}, \delta = \{\delta_1, \delta_2, \ldots, \delta_n\}$ where $\theta, \delta\ \epsilon\ [0,1]$ with $\theta + \delta \leq 1$ then level subset of an intuitionistic multi fuzzy set $\mathcal{A}$ can be defined as;

$(\mathcal{A})_{\theta,\delta} = \{s\ \epsilon\ \mathcal{N} \mid \mathcal{CM}_{\mathcal{A}}(s) \geq \theta\ and\ \mathcal{CN}_{\mathcal{A}}(s) \leq \delta\}$

Proposition 3.18: Let $\mathcal{N}$ be a non-empty set, and A and $\mathcal{B}$ be intuitionistic multi-fuzzy sets of $\mathcal{N}$, then the following results hold;

$(\mathcal{A})_{\theta,\delta} \subseteq (\mathcal{A})_{\gamma,\emptyset}$, if $\theta \geq \gamma$ and $\delta \leq \emptyset$

$$(\mathcal{A})_{1-\delta,\delta} \subseteq (\mathcal{A})_{\theta,\delta} \subseteq (\mathcal{A})_{\theta,1-\theta}$$

$$\mathcal{A} \subseteq \mathcal{B} \Rightarrow (\mathcal{A})_{\theta,\delta} \subseteq (\mathcal{B})_{\theta,\delta}$$

$$(\mathcal{A} \cap \mathcal{B})_{\theta,\delta} = (\mathcal{A})_{\theta,\delta} \cap (\mathcal{B})_{\theta,\delta}$$

$$(\cap \mathcal{A}_i)_{\theta,\delta} = \cap (\mathcal{A}_i)_{\theta,\delta}$$

$$(\mathcal{A})_{0,1} = \mathcal{N}$$

$$(\mathcal{A} \cup \mathcal{B})_{\theta,\delta} \supseteq (\mathcal{A})_{\theta,\delta} \cup (\mathcal{B})_{\theta,\delta}$$

Remark 7. Equality hold for Vii if $\theta + \delta = 1$.

Theorem 3.19: (Sujatha, 2014) Let $(\mathcal{N}, +, \cdot)$ be a NR and $\mathcal{A}$ be an intuitionistic multi-fuzzy sub-NR of $\mathcal{N}$ then $(\mathcal{A})_{\theta,\delta}$ is sub-NR of $\mathcal{N}$ $\forall\, \theta, \delta \in [0,1]$ with $\theta + \delta \leq 1$ and $\mathcal{CM}_{\mathcal{A}}(0) \geq \theta,\ \mathcal{CN}_{\mathcal{A}}(0) \leq \delta$.

Proof. Given that $\mathcal{CM}_{\mathcal{A}}(0) \geq \theta, \mathcal{CN}_{\mathcal{A}}(0) \leq \delta$ and
let $\mathcal{A}$ is an IMF sub $-$ NR of $\mathcal{N}$ and $s, t \in (\mathcal{A})_{\theta,\delta}$

$\Rightarrow \mathcal{CM}_{\mathcal{A}}(s) \geq \theta, \mathcal{CN}_{\mathcal{A}}(s) \leq \delta$ and $\mathcal{CM}_{\mathcal{A}}(t) \geq \theta, \mathcal{CN}_{\mathcal{A}}(t) \leq \delta$

$\Rightarrow \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t) \geq \theta$ and $\mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(s) \leq \delta$

$\Rightarrow \mathcal{CM}_{\mathcal{A}}(s-t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t) \geq \theta$ and $\mathcal{CN}_{\mathcal{A}}(s-t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t) \leq \delta$

Hence $s - t \in (\mathcal{A})_{\theta,\delta}$

Also, $\mathcal{CM}_{\mathcal{A}}(s \cdot t) \geq \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{A}}(t) \geq \theta$ and $\mathcal{CN}_{\mathcal{A}}(s \cdot t) \leq \mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t) \leq \delta$

Hence $s.t \in (\mathcal{A})_{\theta,\delta}$

Theorem 3.20: Let $(\mathcal{N}, +, \cdot)$ be a NR and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$ then $(\mathcal{A})_{\theta,\delta}$ is an ideal of $\mathcal{N}$ $\forall\, \theta, \delta \in [0,1]$ with $\theta + \delta \leq 1$ and $\mathcal{CM}_{\mathcal{A}}(0) \geq \theta,\ \mathcal{CN}_{\mathcal{A}}(0) \leq \delta$.

Proof. Given that $\mathcal{CM}_{\mathcal{A}}(0) \geq \theta, \mathcal{CN}_{\mathcal{A}}(0) \leq \delta$ and
let $\mathcal{A}$ be an $IMFI$ of $\mathcal{N}$ and $s, t \in (\mathcal{A})_{\theta,\delta}$ and $a, b \in \mathcal{N}$

$\Rightarrow \mathcal{CM}_{\mathcal{A}}(s) \geq \theta, \mathcal{CN}_{\mathcal{A}}(s) \leq \delta$ and $\mathcal{CM}_{\mathcal{A}}(t) \geq \theta, \mathcal{CN}_{\mathcal{A}}(t) \leq \delta$. Then we are to show the conditions of Definition 3.2.

$\Rightarrow \mathcal{CM}_{\mathcal{A}}(a + t - a) \geq \mathcal{CM}_{\mathcal{A}}(t) \geq \theta$ and $\mathcal{CN}_{\mathcal{A}}(a + t - a) \leq \mathcal{CN}_{\mathcal{A}}(t) \leq \delta$

Hence $a + t - a \in (\mathcal{A})_{\theta,\delta}$

Also, $\mathcal{CM}_{\mathcal{A}}(a \cdot t) \geq \mathcal{CM}_{\mathcal{A}}(t) \geq \theta$ and $\mathcal{CN}_{\mathcal{A}}(a \cdot t) \leq \mathcal{CN}_{\mathcal{A}}(t) \leq \delta$

Hence $a.t \in (\mathcal{A})_{\theta,\delta}$

Now, $\mathcal{CM}_{\mathcal{A}}((a+t)b - ab) \geq \mathcal{CM}_{\mathcal{A}}(t) \geq \theta$ and $\mathcal{CN}_{\mathcal{A}}\big((a+t)b - ab\big) \leq \mathcal{CN}_{\mathcal{A}}(t) \leq \delta$

Hence $\big((a+t)b - ab\big) \in (\mathcal{A})_{\theta,\delta}$

Definition 3.21: If $\mathcal{A} \times \mathcal{B}$ is the Cartesian product of two intuitionistic fuzzy multi-sets and $\theta = \{\theta_1, \theta_2, \theta_3, \dots, \theta_n\}, \delta = \{\delta_1, \delta_2, \dots, \delta_n\}$ where $\theta, \delta \in [0,1]$ with $\theta + \delta \leq 1$ then level subset of $\mathcal{A} \times \mathcal{B}$ can be defined as;

$(\mathcal{A} \times \mathcal{B})_{\theta,\delta} = \{(s,t) \mid \mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{B}}(t) \geq \theta$ and $\mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{B}}(t) \leq \delta\}$

Proposition 3.22: Let $\mathcal{N}$ be a non-empty set and $\mathcal{A}, \mathcal{B}$ be two intuitionistic fuzzy multisets of $\mathcal{N}$, then $(\mathcal{A} \times \mathcal{B})_{\theta,\delta} = (\mathcal{A})_{\theta,\delta} \times (\mathcal{B})_{\theta,\delta}$

Proof. Let $(s,t) \in (\mathcal{A})_{\theta,\delta} \times (\mathcal{B})_{\theta,\delta} \Leftrightarrow s \in (\mathcal{A})_{\theta,\delta}$ and $t \in (\mathcal{B})_{\theta,\delta} \Leftrightarrow \mathcal{CM}_{\mathcal{A}}(s) \geq \theta$ and $\mathcal{CN}_{\mathcal{A}}(s) \leq \delta$ also, $\mathcal{CM}_{\mathcal{B}}(t) \geq \theta$ and $\mathcal{CN}_{\mathcal{B}}(t) \leq \delta \Leftrightarrow \{\mathcal{CM}_{\mathcal{A}}(s) \wedge \mathcal{CM}_{\mathcal{B}}(t)\} \geq \theta$ *and* $\{\mathcal{CN}_{\mathcal{A}}(s) \vee \mathcal{CN}_{\mathcal{A}}(t)\} \leq \delta \Leftrightarrow (s,t) \in (\mathcal{A} \times \mathcal{B})_{\theta,\delta}$

Hence $(\mathcal{A}\times\mathcal{B})_{\theta,\delta} = (\mathcal{A})_{\theta,\delta}\times(\mathcal{B})_{\theta,\delta}$

Theorem 3.23: Let $(\mathcal{N},+,\cdot)$ be a NR and $\mathcal{A},\mathcal{B}$ be two intuitionistic multi-fuzzy sub-NR of $\mathcal{N}$, and $\mathcal{A}\times\mathcal{B}$ is an intuitionistic multi-fuzzy sub-NR of $\mathcal{N}\times\mathcal{N}$ then $(\mathcal{A}\times\mathcal{B})_{\theta,\delta}$ is sub-NR of $\mathcal{N}\times\mathcal{N}$.

Proof. Let $\mathcal{A}\times\mathcal{B}$ be an IMF sub NR of $\mathcal{N}\times\mathcal{N}$ and $(s_1,t_1),(s_2,t_2)\,\epsilon\,(\mathcal{A}\times\mathcal{B})_{\theta,\delta}$ then;

$\Rightarrow \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1) = \mathcal{CM}_{\mathcal{A}}(s_1)\wedge\mathcal{CM}_{\mathcal{B}}(t_1)\geq\theta$ and $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)$
$= \mathcal{CN}_{\mathcal{A}}(s_1)\vee\mathcal{CN}_{\mathcal{B}}(t_1)\geq\delta$

similarly, $\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_2,t_2) = \mathcal{CM}_{\mathcal{A}}(s_2)\wedge\mathcal{CM}_{\mathcal{B}}(s_2)\geq\theta$ and $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_2,t_2)$
$= \mathcal{CN}_{\mathcal{A}}(s_2)\vee\mathcal{CN}_{\mathcal{B}}(t_2)\geq\delta$

$\Rightarrow \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}\{(s_1,t_1)-(s_2,t_2)\}\geq\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\wedge\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_2,t_2)$
$\geq\theta$ and $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}\{(s_1,t_1)-(s_2,t_2)\}\leq\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\vee\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_2,t_2)$
$\leq\delta$

$\Rightarrow \{(s_1,t_1)-(s_2,t_2)\}\,\epsilon\,(\mathcal{A}\times\mathcal{B})_{\theta,\delta}$

Also, $\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}\{(s_1,t_1)\cdot(s_2,t_2)\}\geq\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\wedge\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_2,t_2)$
$\geq\theta$ and $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}\{(s_1,t_1)\cdot(s_2,t_2)\}\leq\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\vee\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_2,t_2)$
$\leq\delta$

$\Rightarrow \{(s_1,t_1)\cdot(s_2,t_2)\}\,\epsilon\,(\mathcal{A}\times\mathcal{B})_{\theta,\delta}$

Theorem 3.24: Let $(\mathcal{N},+,\cdot)$ be a NR and $\mathcal{A},\mathcal{B}$ be two intuitionistic multi-fuzzy ideals of $\mathcal{N}$ and $\mathcal{A}\times\mathcal{B}$ is an intuitionistic multi-fuzzy ideal of $\mathcal{N}\times\mathcal{N}.\mathcal{Then}, (\mathcal{A}\times\mathcal{B})_{\theta,\delta}$ is an ideal of $\mathcal{N}\times\mathcal{N}$ (Sujatha, 2014).

Proof. Let $\mathcal{A}\times\mathcal{B}$ be an $IMFI$ of $\mathcal{N}\times\mathcal{N}$ and $(s_1,t_1)\epsilon\,(\mathcal{A}\times\mathcal{B})_{\theta,\delta}$ $(a_1,b_1),(a_2,b_2)\,\epsilon\,\mathcal{N}\times\mathcal{N}$ then; $\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\geq\theta, \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\leq\delta$

$\Rightarrow \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}\{(a_1,b_1)+(s_1,t_1)-(a_1,b_1)\}\geq \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\geq\theta$

And, $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}\{(a_1,b_1)+(s_1,t_1)-(a_1,b_1)\}\leq\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\leq\delta$

$\Rightarrow \{(a_1,b_1)+(s_1,t_1)-(a_1,b_1)\}\,\epsilon\,(\mathcal{A}\times\mathcal{B})_{\theta,\delta}$

Also, $\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}\{(a_1,b_1)\cdot(s_1,t_1)\}\geq \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\geq\theta$ and $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}\{(a_1,b_1)\cdot(s_1,t_1)\}\leq$ $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\leq\delta$

$\Rightarrow \{(a_1,b_1)\cdot(s_1,t_1)\}\,\epsilon\,(\mathcal{A}\times\mathcal{B})_{\theta,\delta}$

Also, $\mathcal{CM}_{\mathcal{A}\times\mathcal{B}}\{((a_1,b_1)+(s_1,t_1))(a_2,b_2)-(a_1,b_1)(a_2,b_2)\}\geq \mathcal{CM}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\geq\theta$

And $\mathcal{CN}_{\mathcal{A}\times\mathcal{B}}\{((a_1,b_1)+(s_1,t_1))(a_2,b_2)-(a_1,b_1)(a_2,b_2)\}\leq \mathcal{CN}_{\mathcal{A}\times\mathcal{B}}(s_1,t_1)\leq\delta$

$\Rightarrow \{((a_1,b_1)+(s_1,t_1))(a_2,b_2)-(a_1,b_1)(a_2,b_2)\}\,\epsilon\,(\mathcal{A}\times\mathcal{B})_{\theta,\delta}$

Lemma 3.25: Let $(\mathcal{N},+,\cdot)$ be a NR, and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$ then; $\mathcal{CM}_{\mathcal{A}}(a_1+a_2+\cdots+a_n)\geq\wedge_{1\leq i\leq n}\mathcal{CM}_{\mathcal{A}}(a_i)$ and $\mathcal{CN}_{\mathcal{A}}(a_1+a_2+\cdots+a_n)\leq\vee_{1\leq i\leq n}\mathcal{CN}_{\mathcal{A}}(a_i)\quad\forall\, a_i\,\epsilon\,\mathcal{N}$

Theorem 3.26: Let $(\mathcal{N},+,\cdot)$ be a NR under standard addition and multiplication is defined as $a\cdot b = b\;\forall\, a,b\,\epsilon\mathcal{N}$, and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$. Then, $\mathcal{A}'$is an intuitionistic multi-fuzzy ideal of (matrix near ring) $\mathcal{M}_n(\mathcal{N})$ where $\mathcal{CM}_{\mathcal{A}'}$ and $\mathcal{CN}_{\mathcal{A}'}$ can be defined as; $\mathcal{CM}_{\mathcal{A}'}([a_{ij}]) =$ $\wedge_{1\leq i,j\leq n}\mathcal{CM}_{\mathcal{A}}(a_{ij})$ *and* $\mathcal{CN}_{\mathcal{A}'}([a_{ij}]) = \vee_{1\leq i,j\leq n}\mathcal{CN}_{\mathcal{A}'}(a_{ij})\;\forall\, a_{ij}, b_{ij}\,\epsilon\,\mathcal{M}_n(\mathcal{N})$.

Proof. Let $[a_{ij}],[b_{ij}]\,\epsilon\,\mathcal{M}_n(\mathcal{N})$ then we are to show conditions of Definition 3.2.

$\mathcal{CM}_{\mathcal{A}'}([a_{ij}]-[b_{ij}]) = \wedge_{1\leq i,j\leq n}\mathcal{CM}_{\mathcal{A}}(a_{ij}-b_{ij})\geq\wedge_{1\leq i,j\leq n}\{\mathcal{CM}_{\mathcal{A}}(a_{ij})\wedge$ $\mathcal{CM}_{\mathcal{A}}(b_{ij})\} = \{\wedge_{1\leq i,j\leq n}\mathcal{CM}_{\mathcal{A}}(a_{ij})\}\wedge\{\wedge_{1\leq i,j\leq n}\mathcal{CM}_{\mathcal{A}}(b_{ij})\} = \mathcal{CM}_{\mathcal{A}'}([a_{ij}])\wedge$ $\mathcal{CM}_{\mathcal{A}'}([b_{ij}])$

And $\mathcal{CN}_{\mathcal{A}'}([a_{ij}]-[b_{ij}]) = \vee_{1\le i,j\le n}\mathcal{CN}_{\mathcal{A}}(a_{ij}-b_{ij}) \le \vee_{1\le i,j\le n}\{\mathcal{CN}_{\mathcal{A}}(a_{ij}) \vee \mathcal{CN}_{\mathcal{A}}(b_{ij})\} = \{\vee_{1\le i,j\le n}\mathcal{CN}_{\mathcal{A}}(a_{ij})\} \vee \{\vee_{1\le i,j\le n}\mathcal{CN}_{\mathcal{A}}(b_{ij})\} = \mathcal{CN}_{\mathcal{A}'}([a_{ij}]) \vee \mathcal{CN}_{\mathcal{A}'}([b_{ij}])$

(II) and (IV) of definition 3.2; Since $[a_{ij}].[b_{ij}] = [c_{ij}]$ where $c_{ij} = b_{1j}+b_{2j}+\cdots+b_{nj} \Rightarrow \mathcal{CM}_{\mathcal{A}'}([a_{ij}]\cdot[b_{ij}]) = \mathcal{CM}_{\mathcal{A}'}([c_{ij}]) = \wedge_{1\le i,j\le n}\mathcal{CM}_{\mathcal{A}}(c_{ij}) = \wedge_{1\le i,j\le n}\mathcal{CM}_{\mathcal{A}}(c_{1j})$ but $\mathcal{CM}_{\mathcal{A}}(c_{1j}) \ge \mathcal{CM}_{\mathcal{A}}(b_{1j}) \wedge \mathcal{CM}_{\mathcal{A}}(b_{2j}) \wedge \ldots \wedge \mathcal{CM}_{\mathcal{A}}(b_{nj}) \Rightarrow \mathcal{CM}_{\mathcal{A}'}([a_{ij}]\cdot[b_{ij}]) \ge \mathcal{CM}_{\mathcal{A}}(b_{1j}) \wedge \mathcal{CM}_{\mathcal{A}}(b_{2j}) \wedge \ldots \wedge \mathcal{CM}_{\mathcal{A}}(b_{nj}) = \wedge_{1\le i,j\le n}\mathcal{CM}_{\mathcal{A}}(b_{ij}) = \mathcal{CM}_{\mathcal{A}'}([b_{ij}])$

And $\mathcal{CN}_{\mathcal{A}'}([a_{ij}]\cdot[b_{ij}]) = \mathcal{CN}_{\mathcal{A}'}([c_{ij}]) = \vee_{1\le i,j\le n}\mathcal{CN}_{\mathcal{A}}(c_{ij}) = \vee_{1\le i,j\le n}\mathcal{CN}_{\mathcal{A}}(c_{1j})$ but $\mathcal{CN}_{\mathcal{A}}(c_{1j}) \le \mathcal{CN}_{\mathcal{A}}(b_{1j}) \vee \mathcal{CN}_{\mathcal{A}}(b_{2j}) \vee \ldots \vee \mathcal{CN}_{\mathcal{A}}(b_{nj}) \Rightarrow \mathcal{CN}_{\mathcal{A}'}([a_{ij}]\cdot[b_{ij}]) \le \mathcal{CN}_{\mathcal{A}}(b_{1j}) \vee \mathcal{CN}_{\mathcal{A}}(b_{2j}) \vee \ldots \vee \mathcal{CN}_{\mathcal{A}}(b_{nj}) = \vee_{1\le i,j\le n}\mathcal{CN}_{\mathcal{A}}(b_{1j}) = \mathcal{CN}_{\mathcal{A}'}([b_{ij}])$

Also, $\mathcal{CM}_{\mathcal{A}'}([a_{ij}]+[b_{ij}]-[a_{ij}]) = \wedge_{1\le i,j\le n}\mathcal{CM}_{\mathcal{A}}(a_{ij}+b_{ij}-a_{ij}) \ge \wedge_{1\le i,j\le n}\mathcal{CM}_{\mathcal{A}}(b_{ij}) = \mathcal{CM}_{\mathcal{A}'}([b_{ij}])$ and $\mathcal{CN}_{\mathcal{A}'}([a_{ij}]+[b_{ij}]-[a_{ij}]) = \vee_{1\le i,j\le n}\mathcal{CN}_{\mathcal{A}}(a_{ij}+b_{ij}-a_{ij}) \le \vee_{1\le i,j\le n}\mathcal{CN}_{\mathcal{A}}(b_{ij}) = \mathcal{CN}_{\mathcal{A}'}([b_{ij}])$

Let $[c_{ij}] \epsilon \mathcal{M}_n(\mathcal{N})$ and $\{([a_{ij}]+[b_{ij}])[c_{ij}]-[a_{ij}][c_{ij}]\} = [d_{ij}]$ where $d_{ij} = 0\ \forall\ 1 \le$ i, $j \le$ n then $\mathcal{CM}_{\mathcal{A}'}([d_{ij}]) = \wedge_{1\le i,j\le n}\mathcal{CM}_{\mathcal{A}}(d_{ij}) = \mathcal{CM}_{\mathcal{A}}(0) \ge \mathcal{CM}_{\mathcal{A}}(b_{ij}) = \mathcal{CM}_{\mathcal{A}'}([b_{ij}])$

And $\mathcal{CN}_{\mathcal{A}'}([d_{ij}]) = \vee_{1\le i,j\le n}\mathcal{CN}_{\mathcal{A}}(d_{ij}) = \mathcal{CN}_{\mathcal{A}}(0) \le \mathcal{CN}_{\mathcal{A}}(b_{ij}) = \mathcal{CN}_{\mathcal{A}'}([b_{ij}])$

Example 15. Let $(\mathbb{Z},+,\cdot)$ be a NR and $\mathcal{B}$ be an intuitionistic multi fuzzy ideal of $\mathbb{Z}$ defined in example 10 then

$$\mathcal{CM}\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\right) = \begin{cases}(0.5,0.4) & \text{if } [a_{ij}] \text{ is divisible by 2} \\ (0.4,0.3) & \text{otherwise}\end{cases}$$

$$\mathcal{CN}\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\right) = \begin{cases}(0.5,0.6) & \text{if } [a_{ij}] \text{ is divisible by 2} \\ (0.6,0.7) & \text{otherwise}\end{cases}$$

is an intutionistic multi fuzzy ideal of $\mathcal{M}_2(\mathbb{Z})$.

Theorem 3.27: According to Al-Tahan et. al. (2021), let $(\mathcal{N},+,\cdot)$ be a NR under standard addition and multiplication is defined as $a \cdot b = b\ \forall\ a, b\ \epsilon\ \mathcal{N}$ and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$. Then, $\mathcal{A}'$is an intuitionistic multi-fuzzy ideal of $P_n(\mathcal{N})$. where $\mathcal{CM}_{\mathcal{A}'}$ and $\mathcal{CN}_{\mathcal{A}'}$ can be defined as; $\mathcal{CM}_{\mathcal{A}'}(\hbar_0+\hbar_1 s+\hbar_2 s^2+\cdots+\hbar_n s^n) = \wedge_{0\le i\le n}\mathcal{CM}_{\mathcal{A}}(\hbar_i)$ and $\mathcal{CN}_{\mathcal{A}'}(\hbar_0+\hbar_1 s+\hbar_2 s^2+\cdots+\hbar_n s^n) = \vee_{0\le i\le n}\mathcal{CN}_{\mathcal{A}'}((\hbar_i))\ \forall\ \hbar(s)\ \epsilon\ P_n(\mathcal{N})$.

Proof. Let $\hbar(s), \mathrm{g}(s)\ \epsilon\ P_n(\mathcal{N})$ where $\hbar(s) = \hbar_0+\hbar_1 s+\hbar_2 s^2+\cdots+\hbar_n s^n$, $\mathrm{g}(s) = \mathrm{g}_0+\mathrm{g}_1 s+\mathrm{g}_2 s^2+\cdots+\mathrm{g}_n s^n$ then we are to show conditions of Definition 3.2.

(I) $\mathcal{CM}_{\mathcal{A}'}(\hbar(s)-\mathrm{g}(s)) = \wedge_{0\le i\le n}\mathcal{CM}_{\mathcal{A}}(\hbar_i-\mathrm{g}_i) \ge \wedge_{0\le i\le n}\{\mathcal{CM}_{\mathcal{A}}(\hbar_i) \wedge \mathcal{CM}_{\mathcal{A}}(\mathrm{g}_i)\} = \{\wedge_{0\le i\le n}\mathcal{CM}_{\mathcal{A}}(\hbar_i)\} \wedge \{\wedge_{0\le i\le n}\mathcal{CM}_{\mathcal{A}}(\mathrm{g}_i)\} = \mathcal{CM}_{\mathcal{A}'}(\hbar(s)) - \mathcal{CM}_{\mathcal{A}'}(\mathrm{g}(s))$

And $\mathcal{CN}_{\mathcal{A}'}(\hbar(s)-\mathrm{g}(s)) = \vee_{0\le i\le n}\mathcal{CN}_{\mathcal{A}}(\hbar_i-\mathrm{g}_i) \le \vee_{0\le i\le n}\{\mathcal{CN}_{\mathcal{A}}(\hbar_i) \vee \mathcal{CN}_{\mathcal{A}}(\mathrm{g}_i)\} = \{\vee_{0\le i\le n}\mathcal{CN}_{\mathcal{A}}(\hbar_i)\} \vee \{\vee_{0\le i\le n}\mathcal{CN}_{\mathcal{A}}(\mathrm{g}_i)\} = \mathcal{CN}_{\mathcal{A}'}(\hbar(s)) - \mathcal{CN}_{\mathcal{A}'}(\mathrm{g}(s))$

(II) and (IV) $\hbar(s)\cdot\mathrm{g}(s) = \mathrm{g}(s) \Rightarrow \mathcal{CM}_{\mathcal{A}'}(\hbar(s)\cdot\mathrm{g}(s)) = \mathcal{CM}_{\mathcal{A}'}(\mathrm{g}(s)) \ge \mathcal{CM}_{\mathcal{A}'}(\mathrm{g}(s))$

And $\mathcal{CN}_{\mathcal{A}'}(\hbar(s)\cdot\mathrm{g}(s)) = \mathcal{CN}_{\mathcal{A}'}(\mathrm{g}(s)) \le \mathcal{CN}_{\mathcal{A}'}(\mathrm{g}(s))$

(III), $\mathcal{CM}_{\mathcal{A}'}(\hbar(s) + \mathrm{g}(s) - \hbar(s)) = \wedge_{0\le i\le n}\mathcal{CM}_{\mathcal{A}}(\hbar_i + \mathrm{g}_i - \hbar_i) \ge \wedge_{0\le i\le n}\mathcal{CM}_{\mathcal{A}}(\mathrm{g}_i) = \mathcal{CM}_{\mathcal{A}'}(\mathrm{g}(s))$ and $\mathcal{CN}_{\mathcal{A}'}(\hbar(s) + \mathrm{g}(s) - \hbar(s)) = \vee_{0\le i\le n}\mathcal{CN}_{\mathcal{A}}(\hbar_i + \mathrm{g}_i - \hbar_i) \le \vee_{0\le i\le n}\mathcal{CN}_{\mathcal{A}}(\mathrm{g}_i) = \mathcal{CN}_{\mathcal{A}'}(\mathrm{g}(s))$

(V), Let $u(s) \in P_n(\mathcal{N})$ where $u(s) = u_0 + u_1 s + u_2 s^2 + \cdots + u_n s^n$ and $\left((\hbar(s) + \mathrm{g}(s))u(s) - \hbar(s)\mathrm{g}(s)\right) = 0 \Rightarrow \mathcal{CM}_{\mathcal{A}}(0) \ge \mathcal{CM}_{\mathcal{A}}(u_i) \ge \wedge_{0\le i\le n}\mathcal{CM}_{\mathcal{A}}(u_i) = \mathcal{CM}_{\mathcal{A}'}(u(s))$

And, $\mathcal{CN}_{\mathcal{A}}(0) \le \mathcal{CN}_{\mathcal{A}}(u_i) \le \vee_{0\le i\le n}\mathcal{CN}_{\mathcal{A}}(u_i) = \mathcal{CN}_{\mathcal{A}'}(u(s))$

Example 16. Let $(\mathbb{Z}, +,\cdot)$ be a NR and $\mathcal{B}$ be an intuitionistic multi-fuzzy ideal of $\mathbb{Z}$ defined in example 10 then;

$$\mathcal{CM}(\hbar_0 + \hbar_1 s + \hbar_2 s^2 + \hbar_3 s^3) = \begin{cases} (0.5,0.4) & \text{if } \hbar_0, \hbar_1, \hbar_2, \hbar_3 \text{ are divisible by 2} \\ (0.4,0.3) & \text{otherwise} \end{cases}$$

$$\mathcal{CN}(\hbar_0 + \hbar_1 s + \hbar_2 s^2 + \hbar_3 s^3) = \begin{cases} (0.5,0.6) & \text{if } \hbar_0, \hbar_1, \hbar_2, \hbar_3 \text{ are divisible by 2} \\ (0.6,0.7) & \text{otherwise} \end{cases}$$

It is an intuitionistic multi-fuzzy ideal of $P_3(\mathcal{N}.)$

Proposition 3.28. Let $(\mathcal{N}, +,\cdot)$ be a NR, and $\mathcal{A}$ be an intuitionistic multi-fuzzy ideal of $\mathcal{N}$, then $\mathcal{A}'$is an intuitionistic multi-fuzzy ideal of $P_n(\mathcal{N})$ where $\mathcal{CM}_{\mathcal{A}'}$ and $\mathcal{CN}_{\mathcal{A}'}$ can be defined as; $\mathcal{CM}_{\mathcal{A}'}(\hbar_0 + \hbar_1 s + \hbar_2 s^2 + \cdots + \hbar_n s^n) = \mathcal{CM}_{\mathcal{A}}(\hbar_0)$and $\mathcal{CN}_{\mathcal{A}'}\left(\hbar_0 + \hbar_1 s + \hbar_2 s^2 + \cdots + \hbar_n s^n\right) = \mathcal{CN}_{\mathcal{A}}(\hbar_0)\ \forall\ \hbar(s) \in P_n(\mathcal{N})$.

Proof. Same as theorem 3.27.

5. Conclusion

This study constructed the concept of intuitionistic multi-fuzzy near-rings and intuitionistic multi-fuzzy ideals. It explored and illustrated some properties related to intuitionistic multi-fuzzy near-rings and intuitionistic multi-fuzzy ideals. Moreover, it investigated the support, level subsets and Cartesian product of intuitionistic multi-fuzzy near-rings and ideals. It established results associated with all these new constructions. This work contributes to fuzzy set theory, widely used in multi-criteria decision-making and pattern recognition problems. In the future, one may extend these notions to AI-related decision-making and pattern recognition research. Alternatively, extending to inter-valued fuzzy systems or applying the intuitionistic multi-fuzzy idea to vector spaces and modules is possible.

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